1. (25pts.) A group of \( n \) people witness a crime. Each person can either call the police \((C)\) or not call the police \((N)\) and attaches a value of \( v \) to the police being informed (by anybody) but bears the cost of the call, \( c \). Assume that \( v > c > 0 \). Therefore, each individual receives a payoff of zero if nobody calls, \( v \) if somebody other than him calls, and \( v - c \) if he calls.

(a) Find a pure strategy Nash equilibrium of this game.

**Solution**

The following is a Nash equilibrium: Player 1 calls and the other players do not call. Player 1’s payoff in this action profile is equal to \( v - c \), and is greater than 0, which is the payoff he would get if he deviated and did not call. All the other players get a payoff of \( v \), which is greater than the payoff that they would get if they were to deviate and call: \( v - c \). Therefore, no player has a profitable unilateral deviation and hence this is a Nash equilibrium.

(b) Find the symmetric mixed strategy Nash equilibrium in which each player calls with probability \( p \in (0,1) \).

**Solution**

In a symmetric mixed strategy equilibrium in which both actions are played with positive probability, expected payoffs to calling and not calling must be equal:

\[
v - c = 0 \times \text{prob(no one else calls)} + v \times \text{prob(at least one other person calls)}
\]

or

\[
\frac{c}{v} = \text{prob(no one else calls)}
\]

Let \( p \) be the probability with which each person calls in equilibrium. Then, we must have

\[
\frac{c}{v} = (1 - p)^{n-1}
\]

or

\[
p = 1 - \left(\frac{c}{v}\right)^{1/(n-1)}
\]

2. (25pts.) Two players are involved in a dispute whose settlement can be any real number in \([0,1]\). If the dispute is settled at \( a \), then player 1’s payoff is \(-a\) and player 2’s payoff is \( a \). Player 1 and 2 independently make offers \( x \in [0,1] \) and \( y \in [0,1] \), respectively, and the arbitrator settles by choosing the offer that is closest to his ideal settlement. (Assume that in case of a tie the arbitrator chooses each offer with equal probabilities.) Arbitrator’s ideal settlement \( s \) is not known to player 1 and 2, but both believe that it is a random variable distributed uniformly over \([0,1]\). The two players are expected payoff maximizers and all of the above is common knowledge.

(a) Show that in any Nash equilibrium \( x < y \).

**Solution**

Suppose, first that \( x > y \). Then, expected payoff of player 1 is given by

\[
\left(1 - \frac{x + y}{2}\right)(-x) + \frac{x + y}{2}(-y)
\]

However, she would get a higher payoff \((-y)\) by deviating and choosing \( y \), rather than \( x \). Therefore, in all Nash equilibria it must be the case that \( x \leq y \). Suppose now that \( x = y \) in a Nash equilibrium. If \( x > 0 \), player 1 has a profitable deviation (choosing 0), and if \( x < 1 \), player 2 has a profitable deviation (choosing 1). So, we conclude that in any Nash equilibrium \( x < y \).
(b) Find the set of Nash equilibria of this game.

Solution

For any strategy profile in which \( x < y \), the expected payoffs of the players are given by

\[
U_1(x, y) = \frac{x + y}{2} (-x) + \left( 1 - \frac{x + y}{2} \right) (-y) = -\left( y + \frac{x^2 - y^2}{2} \right)
\]

\[
U_2(x, y) = \frac{x + y}{2} x + \left( 1 - \frac{x + y}{2} \right) y
\]

For any \( x > 0 \), player 1’s expected payoff is strictly decreasing in \( x \), and hence player 1’s Nash equilibrium choice must be 0. Given that \( x = 0 \), player 2’s payoff function is equal to \((1 - y/2)y\), which is maximized at \( y = 1 \). So, if there is a Nash equilibrium, we must have \( x = 0 \) and \( y = 1 \). It is easy to verify (do verify it!) that this is a Nash equilibrium. Therefore, the unique Nash equilibrium of this game is given by \( x = 0, y = 1 \).

3. (25pts.) Consider a two-bidder first-price auction where the value of the object is the same and given by \( v = t_1 + t_2 \) for both bidders. Bidder \( i \) observes the signal \( t_i \) but not the other signal \( t_j, j \neq i \). Signals are distributed independently and uniformly over \([0, 1]\) and in case of a tie the object is won by each player with equal probabilities. Find a symmetric equilibrium of this game where player \( i \)’s strategy is given by \( \beta_i(t_i) = at_i, i = 1, 2, a \geq 0 \).

Solution

First note that, given player 2’s equilibrium strategy, it is never optimal for player 1 to bid strictly larger than \( a \). The expected payoff of player 1 who has type \( t_1 \) and bids \( b \leq a \) is given by

\[
U_1(b, t_1) = E[v - b | b > \beta_2(t_2)] \text{ prob}(b > \beta_2(t_2))
\]

\[
= E[t_1 + t_2 - b | b > at_2] \text{ prob}(b > at_2)
\]

\[
= E[t_1 + t_2 - b | t_2 < b/a] \text{ prob}(t_2 < b/a)
\]

\[
= (t_1 + E[t_2 | t_2 < b/a] - b) \frac{b}{a}
\]

\[
= \left( t_1 + \frac{b}{2a} - b \right) \frac{b}{a}
\]

Suppose the maximum of this function occurs at \( b > 0 \). Then, the first order condition must hold, i.e.,

\[
\frac{\partial U_1(b, t_1)}{\partial b} = (t_1 + \frac{b}{2a} - b) \frac{1}{a} + b \left( \frac{1}{2a} - 1 \right) = 0,
\]

which implies that

\[
2(1 - \frac{1}{2a})b = t_1.
\]

For \( \beta_1 = at_1 \) to be optimal we must have

\[
2(1 - \frac{1}{2a})a t_1 = t_1,
\]

which implies that \( a = 1 \). Therefore, we must have \( \beta_i(t_i) = t_i \) in any such equilibrium.

Conversely, let us show that \( \beta_i(t_i) = t_i \) is indeed an equilibrium. Player 1’s expected payoff to the equilibrium strategy is given by \( t_1^2/2 \). Since bidding greater than 1 can never be a profitable deviation, his expected payoff to any deviation \( b \leq 1 \) is given by \((t_1 - b/2)b\). Therefore, the difference between the equilibrium and the deviation payoffs is given by

\[
\frac{t_1^2}{2} - \left( t_1 - \frac{b}{2} \right) b = \frac{1}{2}(t_1 - b)^2 \geq 0.
\]

Therefore, there is no profitable deviation, proving that this is indeed a Bayesian equilibrium.
4. (25pts.) Let $G = (N, (S_i), (u_i))$ be a finite strategic form game. Show that every pure strategy Nash equilibrium is rationalizable.

**Reminder:** Let $R_i(0) = S_i$ and define

$$R_i(t) = \{ s_i \in R_i(t-1) : \exists \mu_{-i} \in \times_{j \neq i} \Delta(R_j(t-1)), \text{ such that } U_i(s_i, \mu_{-i}) \geq U_i(s_i', \mu_{-i}) \text{ for all } s_i' \in R_i(t-1) \}$$

for $t = 1, 2, \ldots$. The set of rationalizable strategies for player $i$ is $R_i = \bigcap_{t=0}^{\infty} R_i(t)$ and the set of rationalizable outcomes is $R(G) = \times_{i \in N} R_i$.

**Solution**

We will prove the claim by *induction*. We need to prove that if $s^* \in N(G)$, then for any $i \in N$

(a) $s^*_i \in R_i(1)$

(b) $s^*_i \in R_i(k)$ implies that $s^*_i \in R_i(k + 1)$, $k = 1, 2, \ldots$

Let $s^* \in N(G)$ and take any $i \in N$. This implies that $U_i(s^*_i, s^*_{-i}) \geq U_i(s_i, s^*_i)$ for all $s_i \in R_i(0)$. Since $s^*_{-i} \in \times_{j \neq i} \Delta(R_j(0))$, we have $s_i \in R_i(1)$.

Now, suppose $s^*_i \in R_i(k)$ for all $i \in N$. This implies, by definition, that $s^*_i \in \times_{j \neq i} \Delta(R_j(k))$ for all $i \in N$. Since $U_i(s^*_i, s^*_i) \geq U_i(s_i, s^*_i)$ for all $s_i \in S_i$, we have that $s^*_i \in R_i(k + 1)$ for all $i \in N$. Therefore, $s^*_i \in R_i$ for all $i \in N$ and thus $s^* \in R(G)$. 

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