1. (30pts.) Consider the extensive form game given by the following game tree:

![Game Tree Diagram]

(a) (10pts.) Write down the strategic form of this game (The bimatrix representation is sufficient).

Solution

<table>
<thead>
<tr>
<th></th>
<th>l</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ll</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Lr</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Rl</td>
<td>−1,1</td>
<td>0.2</td>
</tr>
<tr>
<td>Rr</td>
<td>−2,2</td>
<td>2.3</td>
</tr>
</tbody>
</table>

(b) (10pts.) Find all pure strategy Nash equilibria.

Solution

The set of pure strategy Nash equilibria is \{(Ll, l), (Lr, l), (Rr, r)\}.

(c) (10pts.) Find the set of pure strategy subgame perfect equilibria of the game.

Solution

The unique pure strategy subgame perfect equilibrium is (Rr, r).

2. (30pts.) An entrepreneur has a project that she presents to a capitalist. She has her own money that she could invest in the project and is looking for additional funding from the capitalist. The project is either good (denoted g) (with probability p) or it is bad (denoted b) (with probability 1 − p) and only the entrepreneur knows the quality of the project. The entrepreneur (E) decides whether to invest her own money (I) or not (N), the capitalist (C) observes whether the entrepreneur has invested or not and then decides whether to invest his money (i) or not (n). Figure 1 represents the game and gives the payoffs, where the first number is the entrepreneur’s payoff and the second number is the capitalist’s.

(a) (20pts.) Find the set of pure strategy perfect Bayesian equilibria of this game.

Solution

First notice that N is strictly dominant for the bad type of the entrepreneur. Therefore, she must be playing N in any PBE, i.e., \(\beta_E(N|b) = 1\). For the good type there are two possibilities:
i. Good type plays $I$: $\beta_E(I|g) = 1$
Bayes’ Law implies that $\mu(g|I) = 1$ and $\mu(b|N) = 1$. This, in turn implies that capitalist plays $i$ after $I$ and $n$ after $N$. Therefore, playing $I$ is sequentially rational for the good type, and the following is a PBE:

$$\beta_E(N|b) = 1, \beta_E(I|g) = 1, \beta_C(i|I) = 1, \beta_C(n|N) = 1, \mu(g|I) = 1, \mu(b|N) = 1$$

ii. Good type plays $N$: $\beta_E(N|g) = 1$
For the good type to play $N$ it must be that the capitalist plays $n$ after $I$, i.e., $\beta_C(n|I) = 1$. This implies that $\mu(g|I) \leq 2/3$. Also, Bayes’ Law implies that $\mu(g|N) = p$ and $\mu(b|N) = 1 - p$. This, in turn, implies that capitalist plays $n$ after $N$, i.e., $\beta_C(n|N) = 1$. Therefore, the following is a PBE:

$$\beta_E(N|b) = 1, \beta_E(N|g) = 1, \beta_C(n|I) = 1, \beta_C(n|N) = 1, \mu(g|N) = p, \mu(g|I) \leq 2/3$$

(b) (10pts.) Check if the equilibria you find in part (a) satisfy the intuitive criterion.

Solution
The equilibrium in which the good type plays $I$ satisfies the intuitive criterion since both types in equilibrium obtain the best that they can in the game. The equilibrium in which the good type plays $N$ does not satisfy the intuitive criterion. To see why, first note that for the bad type action $I$ is dominated by her equilibrium payoff, but this is not the case for the good type. Once we restrict the capitalist’s beliefs to put positive probability only on the good type after the action $I$, his best response is to play $i$. But then playing $I$ gives the good type a strictly higher payoff than her equilibrium payoff.

3. (40pts.) A principal ($P$) needs to make a decision $x \geq 0$ but the optimal decision depends on a state of the world, $t$, which is unknown to her. She believes that $t$ is uniformly distributed over the interval $[0,1]$ and asks an advisor ($A$), who knows the state of the world, to report it. If the state of the world is $t$ and action $x$ is taken, then the principal’s and the advisor’s payoffs are given by

$$u_P(x, t) = -(x - t)^2$$
$$u_A(x, t) = -(x - (t + b))^2$$

where $b \in [0, 1/4]$. Note that the optimal action for the principal is $t$, whereas that for the advisor is $t + b$, i.e., they have conflicting interests over the action. The game has the following sequence of events: (1) The advisor observes $t \in [0, 1]$ and reports a number $r \in [0, 1]$ (not necessarily the true state of the world); (2) the principal observes $r$ (but not $t$) and makes a decision $x \geq 0$, and the game ends. Both players are expected payoff maximizers and all of this is common knowledge.

Hint: For any belief that the principal might hold about $t$, her optimal action is its expected value. Also, given two decisions $x_1$ and $x_2 > x_1$, advisors who have $t + b$ smaller than the mid-point between these decisions, $(x_1 + x_2)/2$, prefer $x_1$, and those who have $t + b$ bigger than the mid-point prefer $x_2$, while the type for whom $t + b = (x_1 + x_2)/2$ is indifferent.
(a) **(10pts.)** Is there a fully revealing perfect Bayesian equilibrium of this game, i.e., a PBE in which for any $t$, the advisor of type $t$ reports $t$?

**Solution**
Suppose that there is a PBE in which the advisor’s strategy is $r(t) = t$. Then, Bayes’ Law implies that the principal’s beliefs put probability one on $r$ upon observing report $r$, and hence her equilibrium strategy is $x(r) = r$. Equilibrium payoff of type $t \in [0,1]$ advisor is $-(t - (t + b))^2 = -b^2$. But type 0 could report $b$, after which the principal would choose $x(b) = b$, and the advisor’s payoff would be $-b^2 = 0 > -b^2$, contradicting that this is a PBE. Therefore, there is no fully revealing perfect Bayesian equilibrium.

(b) **(10pts.)** Find a perfect Bayesian equilibrium of this game in which all the types report 1.

**Solution**
Bayes’ Law implies that the beliefs of the principal after observing the report 1 is uniform over the interval $[0,1]$. Therefore, her optimal strategy is given by $x(1) = 1/2$, which gives an equilibrium payoff of $-1/2 - (t + b))^2$ to type $t$ advisor. For this to be an equilibrium type $t$ advisor must have no profitable deviation, i.e., for any $t \in [0,1]$, $-1/2 - (t + b))^2 \geq -(x(r) - (t + b))^2$ for all $r \in [0,1]$. This is easily satisfied if we specify out-of-equilibrium beliefs to be uniform after any report, since then the equilibrium strategy of the principal is $x(r) = 1/2$ for all $r \in [0,1]$.

(c) **(20pts.)** Find a perfect Bayesian equilibrium of this game in which, for some $t_1 \in (0,1)$, all the types $t < t_1$ report $r_1$ and all the types $t \geq t_1$ report $r_2 \neq r_1$.

**Solution**
Bayes’ Law implies that the beliefs of the principal after observing the report $r_1$ is uniform over the interval $[0,t_1]$ and after report $r_2$, uniform over $[t_1,1]$. Therefore, her optimal strategy is given by $x(r_1) = t_1/2$ and $x(r_2) = (t_1 + 1)/2$. For this to be an equilibrium, type $t$ advisor must have no profitable deviation, for any $t$. In particular all types in $[0,t_1]$ must weakly prefer $t_1/2$ to $(t_1 + 1)/2$ and all types in $[t_1,1]$ must weakly prefer $(t_1 + 1)/2$ to $t_1/2$. This implies that $t_1 + b$ must be equal to the mid-point between $t_1/2$ and $(t_1 + 1)/2$, i.e.,

$$t_1 + b = \frac{1}{2} \left( \frac{t_1}{2} + \frac{t_1 + 1}{2} \right)$$

which is solved as $t_1 = 1/2 - 2b$. Note that, since $b < 1/4$, we have $t_1 \in (0,1)$. Finally, we have to specify out-of-equilibrium beliefs so that no type has an incentive to deviate. This is accomplished by the following beliefs: for any report $r \neq r_2$, the principal believes that $t$ is distributed uniformly over the interval $[0,t_1]$ and after report $r_2$ she believes that $t$ is distributed uniformly over $[t_1,1]$. Therefore, her equilibrium strategy is $x(r) = t_1/2$ for any $r \neq r_2$ and $x(r_2) = (t_1 + 1)/2$. 
