1. **(20pts.)** Find the set of pure strategy subgame perfect equilibria (SPE) of the following game:

![Game Diagram]

**Solution**

The strategic form of the game following $F$ is given by

<table>
<thead>
<tr>
<th></th>
<th>$b$</th>
<th>$s$</th>
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</thead>
<tbody>
<tr>
<td>$b$</td>
<td>2,1</td>
<td>-1,0</td>
</tr>
<tr>
<td>$s$</td>
<td>-1,2</td>
<td>0,1</td>
</tr>
</tbody>
</table>

This game has a unique pure strategy Nash equilibrium: $(b, b)$. The strategic form of the game following $N$ is given by

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>$S$</td>
<td>0,0</td>
<td>1,3</td>
</tr>
</tbody>
</table>

which has two pure strategy Nash equilibria: $(B, B), (S, S)$.

If the equilibrium following $N$ is $(B, B)$, then the optimal action at the beginning of the game for player 1 is $N$. Therefore, the following is a SPE

$(NbB, bB)$

If the equilibrium following $N$ is $(S, S)$, then $F$ is the optimal action. Therefore, the following is another SPE

$(FbS, bS)$

2. **(20pts.)** A firm and a union are bargaining over wages. Total surplus available is $\pi$. The union asks for a wage $w \geq 0$ and the firm accepts ($Y$) or rejects ($N$). If the firm rejects, there is a strike and everybody’s payoff is zero. If the union’s offer of $w$ is accepted, then the payoff of the union is $w$ and that of the firm is $\pi - w$. The firm knows $\pi$ but the union does not. The union believes that $\pi$ is distributed uniformly over the [0,1] interval. Find the subgame perfect equilibria of this game. (Assume that the firm accepts when indifferent).
Solution
The firm accepts if and only if $w \leq \pi$. Therefore, the union’s expected payoff to an offer $w$ is

$$w \cdot \text{prob}(w \leq \pi) = w(1 - \text{prob}(\pi < w))$$

If $w \leq 1$, this is given by

$$w(1 - w)$$

which is maximized at $w = 1/2$, with expected payoff $1/4$. Expected payoff to $w > 1$ is zero. Therefore, the following is the unique SPE (in which the firm accepts when indifferent):

$$w = \frac{1}{2}$$

$$s_F(\pi) = Y \iff w \leq \pi$$

3. (30pts.) Consider the game given in Figure 1, where $a$ is a real number.

(a) (10pts.) Find the set of pure strategy separating perfect Bayesian equilibrium. (Your answer may depend on $a$)

Solution
First suppose $\beta_1(R|X) = 1$ and $\beta_1(L|Y) = 1$. This implies that $\mu(X|R) = 1$ and $\mu(Y|L) = 1$. Therefore, $\beta_2(u|L) = 1$ and $\beta_2(d|R) = 1$. But then type $Y$ would want to deviate. Hence, there is no such PBE.

Now suppose $\beta_1(L|X) = 1$ and $\beta_1(R|Y) = 1$. This implies that $\mu(Y|R) = 1$ and $\mu(X|L) = 1$. Therefore, $\beta_2(d|L) = 1$ and $\beta_2(u|R) = 1$. Type $X$ does not have a profitable deviation and type $Y$ does not have one if and only if $a \leq 2$. So, this is an equilibrium if and only if $a \leq 2$.

(b) (10pts.) Find the set of pure strategy pooling perfect Bayesian equilibrium. (Your answer may depend on $a$)

Solution
First suppose $\beta_1(L|X) = \beta_1(L|Y) = 1$. This implies that $\mu(X|L) = 1/2$, which, in turn, implies that $\beta_2(u|L) = 1$. But then type $Y$ has a profitable deviation irrespective of how player 2 plays following $R$. Thus, there is no such PBE.

Now suppose $\beta_1(R|X) = \beta_1(R|Y) = 1$. This implies that $\mu(X|R) = 1/2$, which, in turn, implies that $\beta_2(u|R) = 1$. The only way neither type has a profitable deviation is if player 2 plays $u$ after $L$, i.e., $\beta_2(u|L) = 1$. Sequential rationality of player 2 after $L$, then, implies that $\mu(X|L) \leq 2/3$. This constitutes a PBE for any value of $a$.

(c) (10pts.) Now suppose that $a = 1$. Find the set of pure strategy equilibria that survive the intuitive criterion.
Solution

The separating equilibrium with $\beta_1(L|X) = 1$ and $\beta_1(R|Y) = 1$ survives the intuitive criterion since there is no out of equilibrium information set. The pooling equilibrium in which $\beta_1(R|X) = \beta_1(R|Y) = 1$, however, does not survive the intuitive criterion. The reason is that in all such equilibria $L$ is equilibrium dominated for type $Y$ but not for type $X$. If we restrict the support of player 2’s beliefs to $X$ after $L$, her best response is to play $d$. Therefore, playing $L$ would bring a higher payoff to type $X$ than his equilibrium payoff of 2.

4. (30pts.) Player 2 will take an action $a \in [0, 1]$ but the optimal action depends on the state of the world, $t$, which is unknown to her. She believes that $t$ is uniformly distributed over the interval $[0, 1]$ and asks player 1, who knows the state of the world, to report it. If the state of the world is $t$ and action $a$ is taken, then the payoffs are given by

$$u_1(a, t) = (1 - a)(1 + ky) + a(t + k)$$

$$u_2(a, t) = a(1 + kt) + (1 - a)(y + k)$$

where $0 < k < 1$ and $1 - k < y < 1$. (Note that the payoffs do not depend on the report).

Find necessary and sufficient conditions for the existence of a perfect Bayesian equilibrium in which, for some $t^* \in (0, 1)$, all the types $t < t^*$ report $r_1$ after which player 2 plays $a_1$, and all the types $t \geq t^*$ report $r_2 \neq r_1$ after which player 2 plays $a_2 \neq a_1$.

Solution

Player 1’s sequential rationality implies that

$$\begin{align*}
(1 - a_1)(1 + ky) + a_2(t + k) &\geq (1 - a_2)(1 + ky) + a_2(t + k), \quad \text{for all } t < t^* \\
(1 - a_2)(1 + ky) + a_2(t + k) &\geq (1 - a_1)(1 + ky) + a_1(t + k), \quad \text{for all } t \geq t^*
\end{align*}$$

or equivalently

$$\begin{align*}
(a_2 - a_1)(1 + ky) &\geq (a_2 - a_1)(t + k), \quad \text{for all } t < t^* \\
(a_2 - a_1)(1 + ky) &\leq (a_2 - a_1)(t + k), \quad \text{for all } t \geq t^*
\end{align*}$$

Since $a_1 \neq a_2$, these conditions imply that $a_2 > a_1$ and that

$$t^* = 1 - k(1 - y)$$

Note that $1 - k(1 - y) \in (0, 1)$ by our assumptions that $0 < k < 1$ and $1 - k < y < 1$.

Bayes’ Law implies that the beliefs of the principal after observing the report $r_1$ is uniform over the interval $[0, t^*)$ and after report $r_2$, uniform over $[t^*, 1]$. This implies that,

$$\begin{align*}
E[t|r_1] &= \frac{1 - k(1 - y)}{2} \\
E[t|r_2] &= \frac{2 - k(1 - y)}{2}
\end{align*}$$

Expected payoff of player 2 after receiving report $r$ is given by

$$a(1 + kE[t|r]) + (1 - a)(y + k)$$

Suppose that $a_2 < 1$. Then, it must be the case that

$$\frac{2 - k(1 - y)}{2} \leq \frac{y}{k} + 1 - \frac{1}{k},$$

which implies that $k^2 \geq 2$, contradicting the assumption that $k < 1$. Therefore, $1 = a_2 > a_1 \geq 0$ and optimality of player 2’s strategy implies that

$$\frac{1 - k(1 - y)}{2} \leq \frac{y}{k} + 1 - \frac{1}{k},$$

which is equivalent to

$$y \geq \frac{2 - k}{2 - k^2}$$
and 
\[ \frac{1 + 1 - k(1 - y)}{2} \geq \frac{y}{k} + 1 - \frac{1}{k}, \]
which is equivalent to \( k^2 \leq 2. \)

Since \( k < 1, \) the only necessary condition therefore is that
\[ y \geq \frac{2 - k}{2 - k^2}. \]

Suppose now that
\[ y \geq \frac{2 - k}{2 - k^2}. \]

Let \( t^* = 1 - k(1 - y) \) and consider the following assessment: all the types \( t < t^* \) report 0 and all the types \( t \geq t^* \) report 1. After report 0 player 2 believes that \( t \) is uniformly distributed over \([0, t^*)\) and plays 0; after any other report she believes that \( t \) is uniformly distributed over \([t^*, 1]\) and plays 1. It is easily verified that this is an equilibrium.