1. **(30pts.)** Consider the extensive form game given by the following game tree:

(a) **(10pts.)** Write down the strategic form of this game (The bimatrix representation is sufficient).

Solution

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(b) **(10pts.)** Find the set of pure strategy Nash equilibria.

Solution

The set of pure strategy Nash equilibria is \{(Ll, r), (Lr, r), (Rl, l)\}.

(c) **(10pts.)** Find the set of (pure and mixed) subgame perfect equilibria.

Solution

The set of Nash equilibria of the subgame after R is

\[ \{(l, l), (r, r), \left( \frac{3}{4}l \oplus \frac{1}{4}r, \frac{1}{4}l \oplus \frac{2}{4}r \right) \}. \]

Therefore, the set of subgame perfect equilibria is

\[ \{(Rl, l), (Lr, r), (L \frac{3}{4}l \oplus \frac{1}{4}r, \frac{1}{4}l \oplus \frac{2}{4}r) \}. \]

2. **(30pts.)** Consider the game given in Figure 1.

(a) **(20pts.)** Find the set of (pure and mixed) strategy perfect Bayesian equilibria.

Solution

First notice that R is strictly dominant for type Y. Therefore, she must be playing R in any PBE, i.e., \( \beta_i(R|Y) = 1 \). For type X there are three possibilities:
3. (40pts.) Two partners are trying to dissolve their partnership. Player 1 currently owns a share of the partnership and player 2 owns $1 - s$. The value of owning the whole partnership is $v_i \in [0, 1]$ for player $i = 1, 2$. Player 1 moves first by choosing a price $p \geq 0$. Player 2 can either buy player 1’s share at $ps$, in which case the payoffs of player 1 and 2 are $ps$ and $v_2 - ps$, respectively, or he may sell his share at $p(1 - s)$, in which case the payoffs are $v_1 - p(1 - s)$. The game can be depicted as in Figure 2.

(a) (15pts.) First assume that $v_1 = v_2 = v$, and that this is common knowledge. What is the subgame perfect equilibrium strategy of player 1?

Solution
We first show that $p = v$ in any SPE. Suppose, for contradiction, that $p < v$ in a SPE. Then, player 2 chooses to buy, which brings a payoff of $ps$ to player 1. If player 1 offers $v + \varepsilon$, where
\[
0 < \varepsilon < \frac{s(v - p)}{1 - s}
\]
player 2 would sell and this would bring player 1 a payoff that is strictly greater that $ps$. Therefore, $p < v$ cannot be a SPE strategy.

(b) (10pts.) Find the perfect Bayesian equilibria that survive the intuitive criterion.

Solution
The equilibrium with $\beta_1(L|X) = 1$ survives the intuitive criterion since there is no out of equilibrium information set. The equilibria in which $\beta_1(R|X) = 1$, however, do not survive the intuitive criterion. The reason is that in all such equilibria $L$ is equilibrium dominated for type $Y$ but not for type $X$. If we restrict the support of player 2’s beliefs to $X$ after $L$, her best response is to play $d$. Therefore, playing $L$ would bring a higher payoff to type $X$ than his equilibrium payoff of 2.

3. (40pts.) Two partners are trying to dissolve their partnership. Player 1 currently owns a share $s \in (0, 1)$ of the partnership and player 2 owns $1 - s$. The value of owning the whole partnership is $v_i \in [0, 1]$ for player $i = 1, 2$. Player 1 moves first by choosing a price $p \geq 0$. Player 2 can either buy player 1’s share at $ps$, in which case the payoffs of player 1 and 2 are $ps$ and $v_2 - ps$, respectively, or he may sell his share at $p(1 - s)$, in which case the payoffs are $v_1 - p(1 - s)$ and $p(1 - s)$. The game can be depicted as in Figure 2.

(a) (15pts.) First assume that $v_1 = v_2 = v$, and that this is common knowledge. What is the subgame perfect equilibrium strategy of player 1?

Solution
We first show that $p = v$ in any SPE. Suppose, for contradiction, that $p < v$ in a SPE. Then, player 2 chooses to buy, which brings a payoff of $ps$ to player 1. If player 1 offers $v + \varepsilon$, where
\[
0 < \varepsilon < \frac{s(v - p)}{1 - s}
\]
player 2 would sell and this would bring player 1 a payoff that is strictly greater that $ps$. Therefore, $p < v$ cannot be a SPE strategy.
Suppose now that \( p > v \) in a SPE. At this price player 2 chooses to sell, which brings player 1 a payoff of \( v - p(1 - s) \). If player 1 offers \( v - \varepsilon \), where

\[
0 < \varepsilon < \min\{v, \frac{(1 - s)(p - v)}{s}\}
\]

player 2 would buy and this would bring player 1 a strictly higher payoff. Therefore, in any SPE \( p = v \). At this price player 2 is indifferent between buying and selling, and whatever he does player 1 gets \( vs \). If player 1 instead offers \( p < v \), her payoff is \( ps < vs \) and if she offers \( p > v \), her payoff is \( v - p(1 - s) < vs \). Therefore, offering \( p = v \) is optimal.

(b) (15pts.) Now assume that \( v_i \) is known only to player \( i = 1, 2 \), but it is common knowledge that each \( v_i \) is independently and uniformly distributed over \([0, 1] \). What is the perfect Bayesian equilibrium strategy of player 1?

Solution

Expected payoff of player 1 to any price offer \( p \) is given by

\[
\text{prob}(v_2 > p)ps + \text{prob}(v_2 < p)(v_1 - p(1 - s)) = (1 - p)ps + p(v_1 - p(1 - s)) = p(v_1 + s - p).
\]

If the maximum occurs at \( p > 0 \), then the first order condition must hold:

\[
v_1 + s - 2p = 0
\]

or

\[
p = \frac{v_1 + s}{2}.
\]

At this price the payoff is strictly positive, whereas the payoff to \( p = 0 \) is zero. Therefore, this is the PBE strategy of player 1.

(c) (10pts.) Assume that the value of the partnership is the same for both players, i.e., \( v_1 = v_2 = v \). However, \( v \) is known only to player 2. Player 1 believes that \( v \) has a uniform distribution over \([0, 1] \) and all of this is common knowledge. What is the perfect Bayesian equilibrium price offer of player 1?

Solution

Expected payoff of player 1 to any price offer \( p \) is given by

\[
\text{prob}(v > p)ps + \text{prob}(v < p)(E[v|v < p] - p(1 - s)) = (1 - p)ps + p\left(\frac{p}{2} - p(1 - s)\right) = p(s - \frac{p}{2}).
\]

If the maximum is in the interior, then the first order condition must hold:

\[
s - p = 0
\]

or \( p = s \). The payoff to this price is positive whereas the payoff to \( p = 0 \) is zero. Therefore, the PBE strategy of player 1 is \( p = s \).