1. (10pts.) Consider a first-price sealed-bid auction in which in case of a tie the winner is determined by a fair coin toss. There are only two bidders for both of whom the value of the object is 1. If a player wins the object by bidding \( b \in \mathbb{R} \), her payoff is equal to \( 1 - b \) and all of this is common knowledge. Formulate this situation as a strategic form game and find its pure strategy Nash equilibria.

**Solution**

The strategic form is given by

- \( N = \{1, 2\} \)
- \( S_i = \mathbb{R}_+, i = 1, 2 \)
- For any \((b_1, b_2)\) and \(i = 1, 2\)

\[
    u_i(b_1, b_2) = \begin{cases} 
    0, & b_i < b_{-i} \\
    (1 - b_i)/2, & b_i = b_{-i} \\
    1 - b_i, & b_i > b_{-i}
    \end{cases}
\]

Both bids must be at most equal to 1. Also, if one of the players bid \( b < 1 \), the other player has no best response. Therefore, both players must bid 1, which is a Nash equilibrium.

2. (30pts.) Consider the same scenario as in Question 1 but assume that the players need to pay an entry fee \( c \in (0, 1) \) in order to make a bid. Players may decide to stay out of the auction, in which case their payoff is zero, or may enter and bid for the object, in which case they have to pay \( c \) whether they win or lose. Assume that players observe the other player’s entry decision only after they make their own entry decision and if only one player enters, then that player wins with any bid.

(a) (10pts.) Show that there is no pure strategy Nash equilibrium of this game.

**Solution**

Suppose that in equilibrium one of the players, say player 1, chooses to stay out with probability 1. The best response of player 2 is to enter and bid 0. But then, player 1 would do better by entering and bidding any amount in \((0, 1 - c)\), contradicting that this is a Nash equilibrium. So, suppose that player 1 enters with probability 1. If player 1 bids \( b_1 < 1 - c \), then player 2 has no best response. If, on the other hand, \( b_1 \geq 1 - c \), best response of player 2 is to stay out. But then, player 1 would rather bid an amount in \((0, 1 - c)\). Therefore, in any Nash equilibrium players must completely mix between entering and staying out.

(b) (10pts.) Find a symmetric mixed strategy Nash equilibrium that satisfies the following conditions:

- Each player enters with probability \( p \in (0, 1) \).
- After entering, each player bids according to a probability distribution function \( F(x) = \text{prob}(\text{bid} \leq x) \). You may assume that \( F \) is continuous everywhere (i.e., does not have an atom), strictly increasing at any \( x \) such that \( F(x) < 1 \), and \( F(0) = 0 \).

**Solution**

Given that the other player enters with probability \( p \) and bids according to \( F \), expected payoff to entering and bidding \( b \) is given by

\[
    -c + (1 - p)(1 - b) + p(1 - b)F(b)
\]

This is negative for any \( b > 1 - c \), which implies that \( F(b) = 1 \) for all \( b \geq 1 - c \). Now suppose that there exists \( b \) such that \( F(b) < 1 \) and

\[
    U(b) = -c + (1 - p)(1 - b) + p(1 - b)F(b) < 0
\]

Since \( U(b) \) is continuous, this implies that there exists an interval around \( b \) such that expected payoff \( U \) is negative for any bid in that interval. Furthermore, since \( F \) is strictly increasing at \( b \), this
interval receives a positive probability under $F$. But then entering with positive probability and bidding according to $F$ cannot be a best response, since shifting the total probability assigned to that interval to not entering would increase expected payoff. Similarly, we cannot have $U(b) > 0$. Therefore, for any $b$ such that $F(b) < 1$

$$-c + (1 - p)(1 - b) + p(1 - b)F(b) = 0$$

Since $F(0) = 0$, we therefore have $1 - p = c$ and

$$F(b) = \frac{cb}{(1 - c)(1 - b)}$$

which is indeed continuous and strictly increasing. Therefore, in any mixed strategy equilibrium $p = c$ and

$$F(b) = \begin{cases} \frac{cb}{(1 - c)(1 - b)}, & b \leq 1 - c \\ 1, & b > 1 - c \end{cases}$$

Conversely, these strategies constitute a Nash equilibrium since each player is indifferent between staying out and entering and bidding any $b \leq 1 - c$, when the other player plays according to this strategy.

**Bonus (10pts.)** Show that the equilibrium bidding strategy $F$ must be continuous (i.e., no bid receives positive probability) and strictly increasing over $[0, 1 - c)$.

**Solution**

Suppose, for contradiction, that in equilibrium there is an $x$ such that prob($b = x$) = $\alpha > 0$ under $F$. Let prob($b < x$) = $\beta$ and note that the expected payoff of a player, say player 1, to entering and bidding $x$ is given by

$$U(x) = -c + (1 - p)(1 - x) + p[\beta(1 - x) + \alpha \frac{1 - x}{2}]$$

We will show that there exists an $0 < \varepsilon < 1 - x$ such that bidding $x + \varepsilon$ brings a strictly higher payoff than $U(x)$, producing the desired contradiction. Note that

$$U(x + \varepsilon) = -c + (1 - p)(1 - x - \varepsilon) + p[(\alpha + \beta)(1 - x - \varepsilon) + \text{prob}(b = x + \varepsilon) \cdot \frac{1 - x - \varepsilon}{2}]$$

Therefore,

$$U(x + \varepsilon) - U(x) \geq \frac{\alpha(1 - p)(1 - x)}{2} - [(1 - p)(\alpha + \beta) + p]\varepsilon$$

Choosing

$$\varepsilon = \frac{\alpha p(1 - x)}{4[p(\alpha + \beta) + (1 - p)]} \in (0, 1 - x)$$

we get $U(x + \varepsilon) - U(x) > 0$. Therefore, player 1 would be better off shifting $\alpha$ to $x + \varepsilon$, contradicting that this is a Nash equilibrium.

Suppose now that there exists $x < y \leq 1 - c$ such that $F(x) = F(y)$. Without loss of generality let $x$ be the smallest such point. Since $F$ is non-decreasing, $F(z) = F(x) = F(y)$ for all $z \in [x, y]$. Again without loss of generality, let $y$ be the largest such point, i.e., $z > y$ implies $F(z) > F(y)$. But then $F$ is strictly increasing at $y$, and hence

$$F(y) = \frac{cy}{(1 - c)(1 - y)} > 0.$$ 

It must be the case that $x > 0$, since $F(0) = 0$ by continuity of $F$. Also, the fact that $x$ is the smallest point at which $F$ is not strictly increasing implies that $F(x) > F(z)$ for all $z < x$, i.e., $F$ is strictly increasing at $x - \varepsilon$ for all $\varepsilon \in (0, x)$. This implies that

$$F(x - \varepsilon) = \frac{c(x - \varepsilon)}{(1 - c)(1 - x + \varepsilon)}$$

for all $\varepsilon \in (0, x)$. Let $G(\varepsilon) \equiv F(x - \varepsilon)$ and note that $G$ is a continuous function of $\varepsilon$ and hence

$$\lim_{\varepsilon \to 0} G(\varepsilon) = G(0) = F(x) = \frac{cx}{(1 - c)(1 - x)} < \frac{cy}{(1 - c)(1 - y)} = F(y)$$

contradicting that $F(x) = F(y)$. 

2
3. (30pts.) Consider the scenario given in Question 2 but assume that players’ values are independently and uniformly distributed over [0,1]. Characterize the symmetric Bayesian equilibria of this game in which each player enters the auction if and only if her value is greater than or equal to some threshold $k$ and bids according to a strategy $\beta : [k,1] \rightarrow \mathbb{R}_+$ that is strictly increasing and continuous everywhere and differentiable over $(k,1)$.

**Solution**

Let us first show that $\beta(k) = 0$. Since, $\beta(k) \geq 0$, the expected payoff of player 1 with value $k$ when she bids 0 is given by

$$-c + \text{prob}(V_2 \leq k)k.$$

Equilibrium bid $\beta(k)$ must yield at least the expected payoff of bidding 0, i.e.,

$$-c + \text{prob}(V_2 \leq \max\{k,\beta^{-1}(\beta(k))\})(k - \beta(k)) = -c + \text{prob}(V_2 \leq k)(k - \beta(k)) \geq -c + \text{prob}(V_2 \leq k)k,$$

which implies that $k = 0$ or $\beta(k) = 0$. Since $k = 0$ implies that the equilibrium payoff of the player with value $k$ is at most $-c$, i.e., smaller than staying out, we must have $\beta(k) = 0$. This, in turn, implies that $\beta^{-1}(b) \geq k$ for all $b \geq 0$.

Therefore, the expected payoff of player 1 with value $v \in [0,1]$ when she bids $b \geq 0$ is given by

$$-c + \text{prob}(V_2 \leq \beta^{-1}(b))(v - b) = -c + \beta^{-1}(b)(v - b).$$

For any $v > k$, we have $\beta(k) > 0$ and hence first order condition must hold with equality, i.e.,

$$\frac{v - \beta(v)}{\beta'(v)} = v = 0.$$

This differential equation can be solved as

$$\beta(v)v = \int_k^v tdt + a$$

for some $a$. Continuity of $\beta$ implies that $\beta(k)k = a$, and $\beta(k) = 0$ implies $a = 0$. Therefore, for any $v \in [k,1]$, we must have

$$\beta(v) = \frac{v^2 - k^2}{2v}.$$

Since a player with value $k$ finds it optimal to enter and bid 0, we must have

$$-c + k^2 \geq 0.$$

Similarly, staying out must be optimal for any player with value $v < k$. In particular, staying out must be at least as good as entering and bidding 0, i.e.,

$$-c + kv \leq 0$$

for all $v < k$, which implies that $c \geq k^2$. Therefore, we obtain $k = \sqrt{c}$. In sum, in any Bayesian equilibrium with the properties stated in the question a player enters if and only if $v \geq \sqrt{c}$ and bids

$$\beta(v) = \frac{v^2 - c}{2v}.$$
4. (30pts.) Consider the following two-player game. Player 1 moves first by choosing either stop (S) or continue (C). If he chooses stop, then the game ends and both players receive a payoff of 1. If he chooses to continue, then player 1 and 2 choose a non-negative integer and receive the product of the two numbers as payoffs. Find the subgame perfect equilibria of this game when

(a) there is no upper bound on the numbers that can be chosen;

**Solution**

Consider the subgame after player 1 chooses to continue. If one of the players chooses a positive integer, then the other player has no best response, whereas everything is a best response to 0. Therefore, the unique Nash equilibrium after player 1 chooses to continue is (0, 0). This implies that player 1’s optimal action at the beginning of the game is to stop.

(b) the maximum integer that can be chosen is $M \geq 2$.

**Solution**

The subgame after player 1 chooses to continue now has another Nash equilibrium given by $(M, M)$, with payoffs equal to $M^2 \geq 4$. Therefore, the game has two subgame perfect equilibria: (1) Player 1 chooses to stop and both choose 0 if player 1 continues; (2) Player 1 chooses to continue and both choose $M$ if player 1 continues.

**Bonus (20pts.)** Consider the strategic form game $G \equiv (N, (S_i, u_i)_{i \in N})$, where $N = \{1, 2\}, S_i = [a, b], b > a$. Define

$$\pi_i(s) \equiv H(s_i, s_1 + s_2)$$

where $H : [a, b] \times [a, 2b] \to \mathbb{R}$ is an arbitrary twice differentiable function with

$$H_1 > 0, H_2 < 0, H_{11} + H_{12} \leq 0$$

and

$$|H_1(a, 2a)| > |H_2(a, 2a)| \text{ and } |H_1(b, 2b)| < |H_2(b, 2b)|.$$ 

The payoff functions are given by

$$u_1(s) = \pi_1(s)$$

$$u_2(s) = F(\pi_2(s), \pi_1(s))$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary differentiable function with $F_1 > 0$ and $F_2 < 0$. Prove that if $s^*$ is a Nash equilibrium of $G$, then $\pi_2(s^*) > \pi_1(s^*)$.

**Solution**

Let $s^*$ be a Nash equilibrium. We will first show that $s^* \neq (a, a)$ and $s^* \neq (b, b)$. Suppose, for contradiction, that $s^* = (a, a)$. Note that $|H_1(a, 2a)| > |H_2(a, 2a)|$ implies

$$\frac{\partial \pi_1(a, a)}{\partial s_1} = H_1(a, 2a) + H_2(a, 2a) > 0.$$ 

This implies that when player 2 is playing $a$, player 1’s best response cannot be $a$. Similarly, $|H_1(b, 2b)| < |H_2(b, 2b)|$, implies that

$$\frac{\partial \pi_1(b, b)}{\partial s_1} = H_1(b, 2b) + H_2(b, 2b) < 0.$$ 

This shows that $(b, b)$ cannot be a Nash equilibrium.

Now suppose $s_1^* = a$ and $s_2^* > a$. Then, $H_1 > 0$ implies that

$$\pi_1(s^*) = H(s_1^*, s_1^* + s_2^*) < H(s_2^*, s_1^* + s_2^*) = \pi_2(s^*),$$

and we are done. Suppose that $s_2^* = b$ and $s_1^* < b$. Then, $\pi_2(s^*) > \pi_1(s^*)$ for the same reason. Therefore, the only cases to consider are when $s_1^* > a$ and $s_2^* < b$. In this case we have the following first order conditions

$$\frac{\partial \pi_1(s^*)}{\partial s_1} \geq 0 \tag{1}$$
and
\[ \frac{\partial u_2(s^*)}{\partial s_2} \leq 0 \] (2)

Note that
\[
\frac{\partial u_2(s^*)}{\partial s_2} = \frac{\partial \pi_2(s^*)}{\partial s_2} F_1 + \frac{\partial \pi_1(s^*)}{\partial s_2} F_2 \\
= \frac{\partial \pi_2(s^*)}{\partial s_2} F_1 + H_2 F_2 
\]

Therefore, \( H_2 < 0, F_2 < 0, F_1 > 0 \), and (2) imply that
\[ \frac{\partial \pi_2(s^*)}{\partial s_2} < 0 \] (3)

Inequalities (1) and (3) imply that
\[ H_1(s_1^*, s_1^* + s_2^*) + H_2(s_1^*, s_1^* + s_2^*) > H_1(s_2^*, s_1^* + s_2^*) + H_2(s_2^*, s_1^* + s_2^*) \]

\( H_{11} + H_{12} \leq 0 \) implies that \( s_1^* < s_2^* \). This, together with \( H_4 > 0 \), implies \( \pi_2(s^*) > \pi_1(s^*) \), and we are done.