Game Theory

Extensive Form Games: Applications

Levent Koçkesen

Koç University
A Simple Game

- You have 10 TL to share
- A makes an offer
  - \(x\) for me and \(10 - x\) for you
- If B accepts
  - A’s offer is implemented
- If B rejects
  - Both get zero
- Half the class will play A (proposer) and half B (responder)
  - Proposers should write how much they offer to give responders
  - I will distribute them randomly to responders
    - They should write Yes or No

Click here for the EXCEL file
Ultimatum Bargaining

- Two players, A and B, bargain over a cake of size 1
- Player A makes an offer $x \in [0, 1]$ to player B
- If player B accepts the offer ($Y$), agreement is reached
  - A receives $x$
  - B receives $1 - x$
- If player B rejects the offer ($N$), both receive zero

\[
\begin{array}{c}
A \\
\downarrow x \\
B \\
\downarrow Y \quad \downarrow N \\
x, 1 - x \quad 0, 0
\end{array}
\]
Subgame Perfect Equilibrium of Ultimatum Bargaining

We can use backward induction

- B’s optimal action
  - $x < 1 \rightarrow$ accept
  - $x = 1 \rightarrow$ accept or reject

1. Suppose in equilibrium B accepts any offer $x \in [0, 1]$
   - What is the optimal offer by A? $x = 1$
   - The following is a SPE

$$x^* = 1$$
$$s_B^*(x) = Y \text{ for all } x \in [0, 1]$$

2. Now suppose that B accepts if and only if $x < 1$
   - What is A’s optimal offer?
     - $x = 1$?
     - $x < 1$?

**Unique SPE**

$$x^* = 1, s_B^*(x) = Y \text{ for all } x \in [0, 1]$$
Bargaining outcomes depend on many factors
  ▶ Social, historical, political, psychological, etc.

Early economists thought the outcome to be indeterminate

John Nash introduced a brilliant alternative approach
  ▶ Axiomatic approach: A solution to a bargaining problem must satisfy certain “reasonable” conditions
    ★ These are the axioms
    ▶ How would such a solution look like?
    ▶ This approach is also known as cooperative game theory

Later non-cooperative game theory helped us identify critical strategic considerations
Bargaining

- Two individuals, A and B, are trying to share a cake of size 1.
- If A gets $x$ and B gets $y$, utilities are $u_A(x)$ and $u_B(y)$.
- If they do not agree, A gets utility $d_A$ and B gets $d_B$.
- What is the most likely outcome?
Bargaining

Let’s simplify the problem

- \( u_A(x) = x \), and \( u_B(x) = x \)
- \( d_A = d_B = 0 \)
- A and B are the same in every other respect
- What is the most likely outcome?
Bargaining

How about now? $d_A = 0.3, d_B = 0.4$

Let $x$ be A’s share. Then

$$\text{Slope} = 1 = \frac{1 - x - 0.4}{x - 0.3}$$

or $x = 0.45$

So A gets 0.45 and B gets 0.55
Bargaining

- In general A gets

\[ d_A + \frac{1}{2}(1 - d_A - d_B) \]

- B gets

\[ d_B + \frac{1}{2}(1 - d_A - d_B) \]

But why is this reasonable?

Two answers:

1. Axiomatic: Nash Bargaining Solution
2. Non-cooperative: Alternating offers bargaining game
Bargaining: Axiomatic Approach

John Nash (1950): The Bargaining Problem, Econometrica

1. Efficiency
   - No waste

2. Symmetry
   - If bargaining problem is symmetric, shares must be equal

3. Scale Invariance
   - Outcome is invariant to linear changes in the payoff scale

4. Independence of Irrelevant Alternatives
   - If you remove alternatives that would not have been chosen, the solution does not change
Nash Bargaining Solution

- What if parties have different bargaining powers?
- Remove symmetry axiom
- Then A gets

\[ x_A = d_A + \alpha(1 - d_A - d_B) \]

- B gets

\[ x_B = d_B + \beta(1 - d_A - d_B) \]

- \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \) represent bargaining powers
- If \( d_A = d_B = 0 \)

\[ x_A = \alpha \quad \text{and} \quad x_B = \beta \]
Alternating Offers Bargaining

- Two players, A and B, bargain over a cake of size 1
- At time 0, A makes an offer $x_A \in [0, 1]$ to B
  - If B accepts, A receives $x_A$ and B receives $1 - x_A$
  - If B rejects, then
- at time 1, B makes a counteroffer $x_B \in [0, 1]$
  - If A accepts, B receives $x_B$ and A receives $1 - x_B$
  - If A rejects, he makes another offer at time 2
- This process continues indefinitely until a player accepts an offer
- If agreement is reached at time $t$ on a partition that gives player $i$ a share $x_i$
  - player $i$’s payoff is $\delta^t_i x_i$
  - $\delta_i \in (0, 1)$ is player $i$’s discount factor
- If players never reach an agreement, then each player’s payoff is zero
\[ \delta_A (1 - x_B), \delta_B x_B \]

\[ \delta_A^2 x_A, \delta_B^2 (1 - x_A) \]
Alternating Offers Bargaining

Stationary No-delay Equilibrium

1. **No Delay**: All equilibrium offers are accepted
2. **Stationarity**: Equilibrium offers do not depend on time

Let equilibrium offers be \((x^*_A, x^*_B)\)

- What does B expect to get if she rejects \(x^*_A\)?
  - \(\delta_B x^*_B\)

Therefore, we must have

\[
1 - x^*_A = \delta_B x^*_B
\]

Similarly

\[
1 - x^*_B = \delta_A x^*_A
\]
Alternating Offers Bargaining

There is a unique solution

\[ x^*_A = \frac{1 - \delta_B}{1 - \delta_A \delta_B} \]
\[ x^*_B = \frac{1 - \delta_A}{1 - \delta_A \delta_B} \]

- There is at most one stationary no-delay SPE
- Still have to verify there exists such an equilibrium
- The following strategy profile is a SPE

Player A: Always offer \( x^*_A \), accept any \( x_B \) with \( 1 - x_B \geq \delta_A x^*_A \)
Player B: Always offer \( x^*_B \), accept any \( x_A \) with \( 1 - x_A \geq \delta_B x^*_B \)
Properties of the Equilibrium

Bargaining Power

Player A’s share

\[ \pi_A = x_A^* = \frac{1 - \delta_B}{1 - \delta_A \delta_B} \]

Player B’s share

\[ \pi_B = 1 - x_A^* = \frac{\delta_B (1 - \delta_A)}{1 - \delta_A \delta_B} \]

- Share of player \( i \) is increasing in \( \delta_i \) and decreasing in \( \delta_j \)
- Bargaining power comes from patience
- Example

\[ \delta_A = 0.9, \delta_B = 0.95 \Rightarrow \pi_A = 0.35, \pi_B = 0.65 \]
Properties of the Equilibrium

First mover advantage

If players are equally patient: \( \delta_A = \delta_B = \delta \)

\[
\pi_A = \frac{1}{1 + \delta} > \frac{\delta}{1 + \delta} = \pi_B
\]

First mover advantage disappears as \( \delta \to 1 \)

\[
\lim_{\delta \to 1} \pi_i = \lim_{\delta \to 1} \pi_B = \frac{1}{2}
\]
Capacity Commitment: Stackelberg Duopoly

- Remember Cournot Duopoly model?
  - Two firms simultaneously choose output (or capacity) levels
  - What happens if one of them moves first?
    - or can commit to a capacity level?
- The resulting model is known as Stackelberg oligopoly
  - After the German economist Heinrich von Stackelberg in Marktform und Gleichgewicht (1934)
- The model is the same except that, now, Firm 1 moves first

Profit function of each firm is given by

\[ u_i(Q_1, Q_2) = (a - b(Q_1 + Q_2))Q_i - cQ_i \]
Nash Equilibrium of Cournot Duopoly

Best response correspondences:

\[ Q_1 = \frac{a - c - bQ_2}{2b} \]
\[ Q_2 = \frac{a - c - bQ_1}{2b} \]

Nash equilibrium:

\( (Q_1^c, Q_2^c) = \left( \frac{a - c}{3b}, \frac{a - c}{3b} \right) \)

In equilibrium each firm’s profit is

\[ \pi_1^c = \pi_2^c = \frac{(a - c)^2}{9b} \]
Cournot Best Response Functions

\[ Q_1^* = \frac{a - c}{b} \]

\[ Q_2^* = \frac{a}{b} \]

\[ B_1 \]

\[ B_2 \]

\[ \frac{a - c}{3b} \]

\[ \frac{a - c}{2b} \]

\[ \frac{a - c}{b} \]
Stackelberg Model

The game has two stages:

1. Firm 1 chooses a capacity level $Q_1 \geq 0$
2. Firm 2 observes Firm 1’s choice and chooses a capacity $Q_2 \geq 0$

$$u_i(Q_1, Q_2) = (a - b(Q_1 + Q_2))Q_i - cQ_i$$
Sequential rationality of Firm 2 implies that for any $Q_1$ it must play a best response:

$$Q_2(Q_1) = \frac{a - c - bQ_1}{2b}$$

Firms 1’s problem is to choose $Q_1$ to maximize:

$$[a - b(Q_1 + Q_2(Q_1))]Q_1 - cQ_1$$
given that Firm 2 will best respond.

Therefore, Firm 1 will choose $Q_1$ to maximize

$$[a - b(Q_1 + \frac{a - c - bQ_1}{2b})]Q_1 - cQ_1$$

This is solved as

$$Q_1 = \frac{a - c}{2b}$$
Backward Induction Equilibrium of Stackelberg Game

Backward Induction Equilibrium

\[ Q_1^s = \frac{a - c}{2b} \]
\[ Q_2^s(Q_1) = \frac{a - c - bQ_1}{2b} \]

Backward Induction Outcome

\[ Q_1^s = \frac{a - c}{2b} > \frac{a - c}{3b} = Q_1^c \]
\[ Q_2^s = \frac{a - c}{4b} < \frac{a - c}{3b} = Q_2^c \]

Equilibrium Profits

\[ \pi_1^s = \frac{(a - c)^2}{8b} > \frac{(a - c)^2}{9b} = \pi_1^c \]
\[ \pi_2^s = \frac{(a - c)^2}{16b} < \frac{(a - c)^2}{9b} = \pi_2^c \]

Firm 1 commits to an aggressive strategy.

There is first mover advantage.