Game Theory
Mixed Strategies

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Matching Pennies

Player 1

\[ \begin{array}{c|cc}
 & H & T \\
\hline
H & -1, 1 & 1, -1 \\
T & 1, -1 & -1, 1 \\
\end{array} \]

How would you play?

No solution?

You should try to be unpredictable

Choose randomly
Drunk Driving

- Chief of police in Istanbul concerned about drunk driving.
- He can set up an alcohol checkpoint or not
  - a checkpoint always catches drunk drivers
  - but costs $c$
- You decide whether to drink wine or cola before driving.
  - Value of wine over cola is $r$
  - Cost of drunk driving is $a$ to you and $f$ to the city
    - incurred only if not caught
  - if you get caught you pay $d$

<table>
<thead>
<tr>
<th></th>
<th>Police Check</th>
<th>Police No</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>You</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wine</td>
<td>$r - d, -c$</td>
<td>$r - a, -f$</td>
</tr>
<tr>
<td>Cola</td>
<td>$0, -c$</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

- Assume: $f > c > 0; d > r > a \geq 0$
Drunk Driving

Let’s work with numbers:

\[ f = 2, \ c = 1, \ d = 4, \ r = 2, \ a = 1 \]

So, the game becomes:

<table>
<thead>
<tr>
<th></th>
<th>Police</th>
<th>Wine</th>
<th>Cola</th>
</tr>
</thead>
<tbody>
<tr>
<td>Check</td>
<td>-2, -1</td>
<td>1, -2</td>
<td>0, -1</td>
</tr>
<tr>
<td>No</td>
<td>1, -2</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

What is the set of Nash equilibria?
A mixed strategy is a probability distribution over the set of actions.

The police chooses to set up checkpoints with probability 1/3. What should you do?

- If you drink cola you get 0
- If you drink wine you get $-2$ with prob. $1/3$ and 1 with prob. $2/3$
  
  What is the value of this to you?
  
  We assume the value is the expected payoff:

\[
\frac{1}{3} \times (-2) + \frac{2}{3} \times 1 = 0
\]

- You are indifferent between Wine and Cola
- You are also indifferent between drinking Wine and Cola with any probability
Mixed Strategy Equilibrium

- You drink wine with probability 1/2
  What should the police do?
  - If he sets up checkpoints he gets expected payoff of $-1$
  - If he does not
    \[
    \frac{1}{2} \times (-2) + \frac{1}{2} \times 0 = -1
    \]
  - The police is indifferent between setting up checkpoints and not, as well as any mixed strategy
- Your strategy is a best response to that of the police and conversely
- We have a **Mixed Strategy Equilibrium**
Mixed Strategy Equilibrium

In a mixed strategy equilibrium every action played with positive probability must be a best response to other players’ mixed strategies.

- In particular players must be indifferent between actions played with positive probability.
- Your probability of drinking wine \( p \)
- The police’s probability of setting up checkpoints \( q \)
- Your expected payoff to
  - Wine is \( q \times (-2) + (1 - q) \times 1 = 1 - 3q \)
  - Cola is 0
- Indifference condition:
  \[
  0 = 1 - 3q
  \]
  implies \( q = 1/3 \)
Mixed Strategy Equilibrium

- The police’s expected payoff to
  - Checkpoint is $-1$
  - Not is $p \times (-2) + (1 - p) \times 0 = -2p$

- Indifference condition

\[ -1 = -2p \]

implies $p = 1/2$

- $(p = 1/2, q = 1/3)$ is a mixed strategy equilibrium

Since there is no pure strategy equilibrium, this is also the unique Nash equilibrium
Hawk-Dove

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
<td>D</td>
</tr>
<tr>
<td>H</td>
<td>0, 0</td>
<td>6, 1</td>
</tr>
<tr>
<td>D</td>
<td>1, 6</td>
<td>3, 3</td>
</tr>
</tbody>
</table>

- How would you play?
- What could be the stable population composition?
- Nash equilibria?
  - $(H, D)$
  - $(D, H)$
- How about $3/4$ hawkish and $1/4$ dovish?
  - On average a dovish player gets $\frac{3}{4} \times 1 + \frac{1}{4} \times 3 = 3/2$
  - A hawkish player gets $\frac{3}{4} \times 0 + \frac{1}{4} \times 6 = 3/2$
  - No type has an evolutionary advantage
- This is a mixed strategy equilibrium
Mixed and Pure Strategy Equilibria

- How do you find the set of all (pure and mixed) Nash equilibria?
- In $2 \times 2$ games we can use the best response correspondences in terms of the mixed strategies and plot them.
- Consider the Battle of the Sexes game

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>$m$</td>
</tr>
<tr>
<td>$m$</td>
<td>2,1</td>
</tr>
<tr>
<td>$o$</td>
<td>0,0</td>
</tr>
</tbody>
</table>

- Denote Player 1’s strategy as $p$ and that of Player 2 as $q$ (probability of choosing $m$)
What is Player 1’s best response?

- Expected payoff to $m$ is $2q$
- Expected payoff to $o$ is $1 - q$

If $2q > 1 - q$ or $q > 1/3$
- best response is $m$ (or equivalently $p = 1$)

If $2q < 1 - q$ or $q < 1/3$
- best response is $o$ (or equivalently $p = 0$)

If $2q = 1 - q$ or $q = 1/3$
- he is indifferent
- best response is any $p \in [0, 1]$

Player 1’s best response correspondence:

$$B_1(q) = \begin{cases} 
\{1\}, & \text{if } q > 1/3 \\
[0, 1], & \text{if } q = 1/3 \\
\{0\}, & \text{if } q < 1/3 
\end{cases}$$
What is Player 2’s best response?

- Expected payoff to
  - $m$ is $p$
  - $o$ is $2(1 - p)$

- If $p > 2(1 - p)$ or $p > 2/3$
  - best response is $m$ (or equivalently $q = 1$)

- If $p < 2(1 - p)$ or $p < 2/3$
  - best response is $o$ (or equivalently $q = 0$)

- If $p = 2(1 - p)$ or $p = 2/3$
  - she is indifferent
  - best response is any $q \in [0, 1]$

Player 2’s best response correspondence:

\[ B_2(p) = \begin{cases} 
\{1\}, & \text{if } p > 2/3 \\
[0, 1], & \text{if } p = 2/3 \\
\{0\}, & \text{if } p < 2/3 
\end{cases} \]
\[ B_1(q) = \begin{cases} 
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\[ B_2(p) = \begin{cases} 
\{1\}, & \text{if } p > 2/3 \\
[0, 1], & \text{if } p = 2/3 \\
\{0\}, & \text{if } p < 2/3 
\end{cases} \]

Set of Nash equilibria
\[ \{(0, 0), (1, 1), (2/3, 1/3)\} \]
Dominated Actions and Mixed Strategies

- Up to now we tested actions only against other actions.
- An action may be undominated by any other action, yet be dominated by a mixed strategy.
- Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1,1</td>
<td>1,0</td>
</tr>
<tr>
<td>M</td>
<td>3,0</td>
<td>0,3</td>
</tr>
<tr>
<td>B</td>
<td>0,1</td>
<td>4,0</td>
</tr>
</tbody>
</table>

- No action dominates T.
- But mixed strategy \((\alpha_1(M) = 1/2, \alpha_1(B) = 1/2)\) strictly dominates T.

A strictly dominated action is never used with positive probability in a mixed strategy equilibrium.
Dominated Actions and Mixed Strategies

- An easy way to figure out dominated actions is to compare expected payoffs.
- Let player 2’s mixed strategy given by \( q = \text{prob}(L) \)

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & 1,1 & 1,0 \\
M & 3,0 & 0,3 \\
B & 0,1 & 4,0 \\
\end{array}
\]

\[
\begin{align*}
\text{u}_1(T, q) &= 1 \\
\text{u}_1(M, q) &= 3q \\
\text{u}_1(B, q) &= 4(1 - q)
\end{align*}
\]

- An action is a never best response if there is no belief (on \( A_{-i} \)) that makes that action a best response.
- \( T \) is a never best response.
- An action is a NBR iff it is strictly dominated.
What if there are no strictly dominated actions?

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<tr>
<td>M</td>
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</tr>
<tr>
<td>B</td>
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</tr>
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Denote player 2’s mixed strategy by $q = \text{prob}(L)$

$u_1(T, q) = 2, u_1(M, q) = 3q, u_1(B, q) = 3(1 - q)$

- Pure strategy Nash eq. $(M, L)$
- Mixed strategy equilibria?
  - Only one player mixes? Not possible
  - Player 1 mixes over $\{T, M, B\}$? Not possible
  - Player 1 mixes over $\{M, B\}$? Not possible
  - Player 1 mixes over $\{T, B\}$? Let $p = \text{prob}(T)$
    - $q = 1/3, 1 - p = p \rightarrow p = 1/2$
  - Player 1 mixes over $\{T, M\}$? Let $p = \text{prob}(T)$
    - $q = 2/3, 3(1 - p) = p \rightarrow p = 3/4$
Real Life Examples?

- **Ignacio Palacios-Huerta (2003):** 5 years’ worth of penalty kicks
- **Empirical scoring probabilities**

\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & 58, 42 & 95, 5 \\
R & 93, 7 & 70, 30 \\
\end{array}
\]

\( R \) is the natural side of the kicker

- **What are the equilibrium strategies?**
Penalty Kick

\[
\begin{array}{cc}
L & R \\
L & 58, 42 & 95, 5 \\
R & 93, 7 & 70, 30 \\
\end{array}
\]

- Kicker must be indifferent

\[58p + 95(1 - p) = 93p + 70(1 - p) \Rightarrow p = 0.42\]

- Goal keeper must be indifferent

\[42q + 7(1 - q) = 5q + 30(1 - q) \Rightarrow q = 0.39\]

<table>
<thead>
<tr>
<th></th>
<th>Theory</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kicker</td>
<td>39%</td>
<td>40%</td>
</tr>
<tr>
<td>Goallie</td>
<td>42%</td>
<td>42%</td>
</tr>
</tbody>
</table>

- Also see Walker and Wooders (2001): Wimbledon