Game Theory
Strategic Form Games

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Strategic Form Games

- It is used to model situations in which players choose strategies without knowing the strategy choices of the other players
- Also known as normal form games

A strategic form game is composed of

1. Set of players: $N$
2. A set of actions: $A_i$ for each player $i$
3. A payoff function: $u_i : A \rightarrow \mathbb{R}$ for each player $i$

$$G = (N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$$

- An outcome $a = (a_1, ..., a_n)$ is a collection of actions, one for each player
  - Also known as an action profile or strategy profile
- outcome space
  $$A = \{(a_1, ..., a_n) : a_i \in A_i, i = 1, ..., n\}$$
Payoff Functions

- Payoff functions represent preferences over the set of outcomes.
- They are ordinal (for now).
- Remember Prisoners’ Dilemma.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>(−5, −5)</td>
</tr>
<tr>
<td></td>
<td>(−6, 0)</td>
</tr>
</tbody>
</table>

The following also represents the same game whenever $a < b < c < d$.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>(b, b)</td>
</tr>
<tr>
<td></td>
<td>(a, d)</td>
</tr>
</tbody>
</table>
Contribution Game

- Everybody starts with 10 TL
- You decide how much of 10 TL to contribute to joint fund
- Amount you contribute will be doubled and then divided equally among everyone
- I will distribute slips of paper that looks like this

Name: ______________

Your Contribution: ______________

- Write your name and an integer between 0 and 10
- We will collect them and enter into Excel
- We will choose one player randomly and pay her

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Example: Price Competition

- Toys “R” Us and Wal-Mart have to decide whether to sell a particular toy at a high or low price.
- They act independently and without knowing the choice of the other store.
- We can write this game in a bimatrix format.

<table>
<thead>
<tr>
<th></th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>High</strong></td>
<td>10, 10</td>
<td>2, 15</td>
</tr>
<tr>
<td><strong>Low</strong></td>
<td>15, 2</td>
<td>5, 5</td>
</tr>
</tbody>
</table>
Example: Price Competition

\begin{tabular}{|c|c|c|}
\hline
 & \(H\) & \(L\) \\
\hline
\(T\) & 10, 10 & 2, 15 \\
\hline
\(W\) & 15, 2 & 5, 5 \\
\hline
\end{tabular}

- \(N = \{T, W\}\)
- \(A_T = A_W = \{H, L\}\)
- \(u_T(H, H) = 10\)
- \(u_W(H, L) = 15\)
- etc.

- What should Toys “R” Us play?
- Does that depend on what it thinks Wal-Mart will do?
- Low is an example of a dominant strategy
- it is optimal independent of what other players do
- How about Wal-Mart?
- (Low, Low) is a dominant strategy equilibrium
Dominant Strategies

- \( a_{-i} = \) profile of actions taken by all players other than \( i \)
- \( A_{-i} = \) the set of all such profiles

An action \( a_i \) strictly dominates \( b_i \) if

\[
    u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}
\]

\( a_i \) weakly dominates action \( b_i \) if

\[
    u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}
\]

and

\[
    u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \quad \text{for some } a_{-i} \in A_{-i}
\]

An action \( a_i \) is strictly dominant if it strictly dominates every action in \( A_i \). It is called weakly dominant if it weakly dominates every action in \( A_i \).
If every player has a (strictly or weakly) dominant strategy, then the corresponding outcome is a (strictly or weakly) dominant strategy equilibrium.

- $L$ strictly dominates $H$
- $(L,L)$ is a strictly dominant strategy equilibrium

- $L$ weakly dominates $H$
- $(L,L)$ is a weakly dominant strategy equilibrium
A reasonable solution concept
It only demands the players to be rational
It does not require them to know that the others are rational too
But it does not exist in many interesting games
Guess the Average

- We will play a game
- I will distribute slips of paper that looks like this

Round 1
Name: _______________
Your guess: _______________

- Write your name and a number between 0 and 100
- We will collect them and enter into Excel
- The number that is closest to half the average wins
- Winner gets 5TL (in case of a tie we choose randomly)

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Price Matching

- Toys “R” Us web page has the following advertisement

- Sounds like a good deal for customers
- How does this change the game?
Price Matching

<table>
<thead>
<tr>
<th>Toys “R” us</th>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>High</strong></td>
<td>10, 10</td>
</tr>
<tr>
<td><strong>Low</strong></td>
<td>15, 2</td>
</tr>
<tr>
<td><strong>Match</strong></td>
<td>10, 10</td>
</tr>
</tbody>
</table>

- Is there a dominant strategy for any of the players?
- There is no dominant strategy equilibrium for this game.
- So, what can we say about this game?
Price Matching

<table>
<thead>
<tr>
<th></th>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
</tr>
<tr>
<td><strong>High</strong></td>
<td>10, 10</td>
</tr>
<tr>
<td><strong>Low</strong></td>
<td>15, 2</td>
</tr>
<tr>
<td><strong>Match</strong></td>
<td>10, 10</td>
</tr>
</tbody>
</table>

- High is weakly dominated and Toys “R” us is rational
  - Toys “R” us should not use High
- High is weakly dominated and Wal-Mart is rational
  - Wal-Mart should not use High
- Each knows the other is rational
  - Toys “R” us knows that Wal-Mart will not use High
  - Wal-Mart knows that Toys “R” us will not use High
  - This is where we use common knowledge of rationality
Price Matching

Therefore we have the following “effective” game

<table>
<thead>
<tr>
<th></th>
<th>Wal-Mart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>5, 5</td>
</tr>
<tr>
<td>Match</td>
<td>5, 5</td>
</tr>
<tr>
<td></td>
<td>5, 5</td>
</tr>
<tr>
<td></td>
<td>10, 10</td>
</tr>
</tbody>
</table>

Low becomes a weakly dominated strategy for both

Both companies will play Match and the prices will be high

The above procedure is known as Iterated Elimination of Dominated Strategies (IEDS)

To be a good strategist try to see the world from the perspective of your rivals and understand that they will most likely do the same
Dominated Strategies

- A “rational” player should never play an action when there is another action that gives her a higher payoff irrespective of how the others play.
- We call such an action a dominated action.

An action $a_i$ is strictly dominated by $b_i$ if

$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i})$$

for all $a_{-i} \in A_{-i}$.

$a_i$ is weakly dominated by $b_i$ if

$$u_i(a_i, a_{-i}) \leq u_i(b_i, a_{-i})$$

for all $a_{-i} \in A_{-i}$

while

$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i})$$

for some $a_{-i} \in A_{-i}$.
Iterated Elimination of Dominated Strategies

- Common knowledge of rationality justifies eliminating dominated strategies iteratively
- This procedure is known as **Iterated Elimination of Dominated Strategies**
- If every strategy eliminated is a strictly dominated strategy
  - **Iterated Elimination of Strictly Dominated Strategies**
- If at least one strategy eliminated is a weakly dominated strategy
  - **Iterated Elimination of Weakly Dominated Strategies**
IESDS vs. IEWDS

- Order of elimination does not matter in IESDS
- It matters in IEWDS

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 3,1 & 2,0 \\
M & 4,0 & 1,1 \\
D & 4,4 & 2,4 \\
\end{array}
\]

- Start with \(U\)
- Start with \(M\)
Effort Game

- You choose how much effort to expend for a joint project
  - An integer between 1 and 7
- The quality of the project depends on the smallest effort: $e$
  - Weakest link
- Effort is costly
- If you choose $e$ your payoff is

$$6 + 2e - e$$

- We will play this twice
- We will randomly choose one round and one student and pay her
- Enter your name and effort choice for Round 1
  Click here for the EXCEL file
Game of Chicken

- There are two providers of satellite radio: XM and Sirius
- XM is the industry leader with 5 million subscribers; Sirius has 2.2 million
- In the long-run the market can sustain only one provider

<table>
<thead>
<tr>
<th></th>
<th>Stay</th>
<th>Exit</th>
</tr>
</thead>
<tbody>
<tr>
<td>XM</td>
<td>-200, -200</td>
<td>300, 0</td>
</tr>
<tr>
<td>Exit</td>
<td>0, 300</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- Is there a dominated strategy?
- What are the likely outcomes?
- Could (Stay, Stay) be an outcome?
- If XM expects Sirius to exit, what is its best strategy (best response)?
- If Sirius expects XM to stay what is its best response?
- (Stay, Exit) is an outcome such that
  - Each player best responds, given what she believes the other will do
  - Their beliefs are correct
- It is a Nash equilibrium.
Nash Equilibrium

- Nash equilibrium is a strategy profile (a collection of strategies, one for each player) such that each strategy is a best response (maximizes payoff) to all the other strategies.

An outcome $a^* = (a_1^*, ..., a_n^*)$ is a **Nash equilibrium** if for each player $i$

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \text{for all } a_i \in A_i$$

- Nash equilibrium is self-enforcing: no player has an incentive to deviate unilaterally.
- One way to find Nash equilibrium is to first find the **best response correspondence** for each player.
  - Best response correspondence gives the set of payoff maximizing strategies for each strategy profile of the other players.
- ... and then find where they “intersect.”
Nash Equilibrium

- XM’s best response to Stay is Exit
- Its best response to Exit is Stay
- Sirius’ best response to Stay is Exit and to Exit is Stay
- Best response correspondences intersect at (Stay, Exit) and (Exit, Stay)
- These two strategy profiles are the two Nash equilibria of this game
- We would expect one of the companies to exit in the long-run

<table>
<thead>
<tr>
<th></th>
<th>XM</th>
<th>Sirius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stay</td>
<td>-200, -200</td>
<td>300, 0</td>
</tr>
<tr>
<td>Exit</td>
<td>0, 300</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
The best response correspondence of player $i$ is given by

$$B_i(a_{-i}) = \{ a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \text{ for all } b_i \in A_i \}.$$ 

$B_i(a_{-i})$ is a set and may not be a singleton in the XM-Sirius game.

<table>
<thead>
<tr>
<th>XM</th>
<th>Stay</th>
<th>Exit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stay</td>
<td>$-200$, $-200$</td>
<td>$300$, $0$</td>
</tr>
<tr>
<td>Exit</td>
<td>$0$, $300$</td>
<td>$0$, $0$</td>
</tr>
</tbody>
</table>

$B_X(\text{Stay}) = \text{Exit}$  \quad  $B_X(\text{Exit}) = \text{Stay}$

$B_S(\text{Stay}) = \text{Exit}$  \quad  $B_S(\text{Exit}) = \text{Stay}$
Stag Hunt

Jean-Jacques Rousseau in *A Discourse on Inequality*

*If it was a matter of hunting a deer, everyone well realized that he must remain faithful to his post; but if a hare happened to pass within reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple...*

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>2,2</td>
<td>0,1</td>
</tr>
<tr>
<td>Hare</td>
<td>1,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Does it look like a game we have seen before?

<table>
<thead>
<tr>
<th></th>
<th>Bonds</th>
<th>Venture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td>110,110</td>
<td>110,100</td>
</tr>
<tr>
<td></td>
<td>100,110</td>
<td>120,120</td>
</tr>
</tbody>
</table>

How would you play these games?
Stag Hunt

- Set of Nash equilibria:

\[ N(SH) = \{(S, S), (H, H)\} \]

\[ N(IG) = \{(Bonds, Bonds), (Venture, Venture)\} \]

- What do you think?
Nash Demand Game

- Each of you will be randomly matched with another student
- You are trying to divide 10 TL
- Each writes independently how much she wants (in multiples of 1 TL)
- If two numbers add up to greater than 10 TL each gets nothing
- Otherwise each gets how much she wrote
- Write your name and demand on the slips
- I will match two randomly
- Choose one pair randomly and pay them

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Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $D \subset \mathbb{R}^n$. A constrained optimization problem is

$$\max f(x) \text{ subject to } x \in D$$

- $f$ is the objective function
- $D$ is the constraint set
- A solution to this problem is $x \in D$ such that

$$f(x) \geq f(y) \text{ for all } y \in D$$

Such an $x$ is called a maximizer

- The set of maximizers is denoted

$$\arg\max \{f(x) | x \in D\}$$

- Similarly for minimization problems
A Graphical Example
Example

\[
\text{max } x^3 - 3x^2 + 2x + 1 \quad \text{subject to} \quad 0.1 \leq x \leq 2.5
\]
Example

$$\max -(x - 1)^2 + 2 \text{ s.t. } x \in [0, 2].$$
A Simple Case

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the problem $\max_{x \in [a,b]} f(x)$.

We call a point $x^*$ such that $f'(x^*) = 0$ a critical point.

\[ f'(x^*) = 0 \]

\[ f'(x^{**}) \neq 0 \]
**Theorem**

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $a < x^* < b$ is a local maximum (minimum) of $f$ on $[a, b]$. Then, $f'(x^*) = 0$.

- Known as first order conditions
- Only necessary for interior local optima
  - Not necessary for global optima
  - Not sufficient for local optima.
- To distinguish between interior local maximum and minimum you can use second order conditions

**Theorem**

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $a < x^* < b$ is a local maximum (minimum) of $f$ on $[a, b]$. Then, $f''(x^*) \leq 0$ ($f''(x^*) \geq 0$).
Recipe for solving the simple case

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable function and consider the problem \( \max_{x \in [a,b]} f(x) \). If the problem has a solution, then it can be found by the following method:

1. Find all critical points: i.e., \( x^* \in [a, b] \) s.t. \( f'(x^*) = 0 \)
2. Evaluate \( f \) at all critical points and at boundaries \( a \) and \( b \)
3. The one that gives the highest \( f \) is the solution

- We can use Weierstrass theorem to determine if there is a solution
- Note that if \( f'(a) > 0 \) (or \( f'(b) < 0 \)), then the solution cannot be at \( a \) (or \( b \))
Example

\[ \max x^2 \text{ s.t. } x \in [-1, 2]. \]

Solution

\[ x^2 \text{ is continuous and } [-1, 2] \text{ is closed and bounded, and hence compact. Therefore, by Weierstrass theorem the problem has a solution.} \]

\[ f'(x) = 2x = 0 \text{ is solved at } x = 0, \text{ which is the only critical point. We have } f(0) = 0, f(-1) = 1, f(2) = 4. \text{ Therefore, 2 is the global maximum.} \]
Example

\[ \max - (x - 1)^2 + 2 \text{ s.t. } x \in [0, 2]. \]

Solution

\[ f \text{ is continuous and } [0, 2] \text{ is compact. Therefore, the problem has a solution. } f'(x) = -2(x - 1) = 0 \text{ is solved at } x = 1, \text{ which is the only critical point. We have } f(1) = 2, f(0) = 1, f(2) = 1. \text{ Therefore, } 1 \text{ is the global maximum. Note that } f'(0) > 0 \text{ and } f'(2) < 0 \text{ and hence we could have eliminated } 0 \text{ and } 2 \text{ as candidates.} \]

What is the solution if the constraint set is \([-1, 0.5]\)?
Recipe for general problems

- Generalizes to \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and the problem is

\[
\max f(x) \quad \text{subject to } x \in \mathcal{D}
\]

- Find critical points \( x^* \in \mathcal{D} \) such that \( Df(x^*) = 0 \)
- Evaluate \( f \) at the critical points and the boundaries of \( \mathcal{D} \)
- Choose the one that gives the highest \( f \)

Important to remember that solution must exist for this method to work

In more complicated problems evaluating \( f \) at the boundaries could be difficult

For such cases we have the method of the Lagrangean (for equality constraints) and Kuhn-Tucker conditions (for inequality constraints)
Cournot Duopoly

- Two firms competing by choosing how much to produce
- Augustine Cournot (1838)

Inverse demand function

\[ p(q_1 + q_2) = \begin{cases} 
  a - b(q_1 + q_2), & q_1 + q_2 \leq a/b \\
  0, & q_1 + q_2 > a/b 
\end{cases} \]

Cost function of firm \( i = 1, 2 \)

\[ c_i(q_i) = cq_i \]

where \( a > c \geq 0 \) and \( b > 0 \)

Therefore, payoff function of firm \( i = 1, 2 \) is given by

\[ u_i(q_1, q_2) = \begin{cases} 
  (a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq a/b \\
  -cq_i, & q_1 + q_2 > a/b 
\end{cases} \]
**Claim**

*Best response correspondence of firm $i \neq j$ is given by*

$$B_i(q_j) = \begin{cases} \frac{a-c-bq_j}{2b}, & q_j < \frac{a-c}{b} \\ 0, & q_j \geq \frac{a-c}{b} \end{cases}$$

**Proof.**

- If $q_2 \geq \frac{a-c}{b}$, then $u_1(q_1, q_2) < 0$ for any $q_1 > 0$. Therefore, $q_1 = 0$ is the unique payoff maximizer.

- If $q_2 < \frac{a-c}{b}$, then the best response cannot be $q_1 = 0$ (why?). Furthermore, it must be the case that $q_1 + q_2 \leq \frac{a-c}{b} \leq \frac{a}{b}$, for otherwise $u_1(q_1, q_2) < 0$. So, the following first order condition must hold

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - 2bq_1 - bq_2 = 0$$

Similarly for firm 2.
Claim

The set of Nash equilibria of the Cournot duopoly game is given by

\[ N(G) = \left\{ \left( \frac{a - c}{3b}, \frac{a - c}{3b} \right) \right\} \]

Proof.

Suppose \((q_1^*, q_2^*)\) is a Nash equilibrium and \(q_i^* = 0\). Then,

\[ q_j^* = \frac{(a - c)}{2b} < \frac{(a - c)}{b}. \]

But, then \(q_i^* \notin B_i(q_j^*)\), a contradiction. Therefore, we must have \(0 < q_i^* < \frac{(a - c)}{b}\), for \(i = 1, 2\). The rest follows from the best response correspondences.
Cournot Nash Equilibrium

\[ a - c \]

\[ B_1(q_2) \]

\[ B_2(q_1) \]

Nash Equilibrium

\[ \frac{a - c}{b} \]

\[ \frac{a - c}{2b} \]

\[ \frac{a - c}{3b} \]
Cournot Oligopoly

In equilibrium each firm’s profit is

\[
\frac{(a - c)^2}{9b}
\]

- Is there a way for these two firms to increase profits?
- What if they form a cartel?
  - They will maximize
    \[
    U(q_1 + q_2) = (a - c - b(q_1 + q_2))(q_1 + q_2)
    \]
  - Optimal level of total production is
    \[
    q_1 + q_2 = \frac{a - c}{2b}
    \]
  - Half of the maximum total profit is
    \[
    \frac{(a - c)^2}{8b}
    \]
- Is the cartel stable?