Game Theory

Problem Set 5 Solutions

Levent Koçkesen

1. Find all the pure and mixed strategy equilibria of the following games by constructing the best response correspondences of the players:

(a) Matching Pennies:

\[
\begin{array}{c|cc}
\text{H} & 1 & -1 \\
\hline
1 & -1 & 1 \\
T & -1 & 1 \\
\end{array}
\]

Let \( \alpha_1 (H) = p \) and \( \alpha_2 (H) = q \). The expected payoff of player 1 to actions \( H \) and \( T \) are

\[
\begin{align*}
  u_1 (H, q) &= 1 \times q + (-1) \times (1 - q) \\
             &= 2q - 1 \\
  u_1 (T, q) &= (-1) \times q + 1 \times (1 - q) \\
             &= 1 - 2q
\end{align*}
\]

Therefore, player 1 prefers to play \( H \) if \( 2q - 1 > 1 - 2q \) or \( q > 1/2 \), and prefers to play \( T \) if \( q < 1/2 \). If \( q = 1/2 \) she is indifferent between \( H \) and \( T \). In other words,

\[
B_1 (q) = \begin{cases} 
  \{0\}, & \text{if } q < 1/2 \\
  [0, 1], & \text{if } q = 1/2 \\
  \{1\}, & \text{if } q > 1/2
\end{cases}
\]

where \( B_1 (q) \) is the best response of player 1 in terms of the choice of \( p \).

Similarly, for player 2

\[
\begin{align*}
  u_2 (p, H) &= (-1) \times p + 1 \times (1 - p) \\
             &= 1 - 2p \\
  u_2 (p, T) &= 1 \times p + (-1) \times (1 - p) \\
             &= 2p - 1
\end{align*}
\]

and hence

\[
B_2 (p) = \begin{cases} 
  \{1\}, & \text{if } p < 1/2 \\
  [0, 1], & \text{if } p = 1/2 \\
  \{0\}, & \text{if } p > 1/2
\end{cases}
\]
Plotting the best response correspondences, you may verify that this game has a unique Nash equilibrium given by \( p^* = q^* = 1/2 \).

(b) Hawk-Dove:

\[
\begin{array}{cc}
H & D \\
H & [0, 0] & [6, 1] \\
D & [1, 6] & [3, 3] \\
\end{array}
\]

Let \( \alpha_1 (H) = p \) and \( \alpha_2 (H) = q \). The expected payoff of player 1 to actions \( H \) and \( D \) are

\[
\begin{align*}
u_1 (H, q) &= 0 \times q + 6 \times (1 - q) \\
\quad &= 6 (1 - q) \\
u_1 (D, q) &= 1 \times q + 3 \times (1 - q) \\
\quad &= 3 - 2q
\end{align*}
\]

Therefore, player 1 prefers to play \( H \) if \( 6 (1 - q) > 3 - 2q \) or \( q < 3/4 \), and prefers to play \( D \) if \( q > 3/4 \). If \( q = 3/4 \) she is indifferent between \( H \) and \( D \). In other words,

\[
B_1 (q) = \begin{cases} 
\{1\}, & \text{if } q < 3/4 \\
[0, 1], & \text{if } q = 3/4 \\
\{0\}, & \text{if } q > 3/4 
\end{cases}
\]

where \( B_1 (q) \) is the best response of player 1 in terms of the choice of \( p \).

Similarly, for player 2

\[
B_2 (p) = \begin{cases} 
\{1\}, & \text{if } p < 3/4 \\
[0, 1], & \text{if } p = 3/4 \\
\{0\}, & \text{if } p > 3/4 
\end{cases}
\]

Therefore, this game has three Nash equilibria (plot the best response correspondences): two are pure strategy equilibria given by \( p = 0, q = 1 \) and \( p = 1, q = 0 \), and the third one is a mixed strategy equilibrium given by \( p = q = 3/4 \).

2. (A Coordination Game). Two people can perform a task if, and only if, they both work. The cost of effort is \( 0 < c < 1 \), and if the task is performed their payoff is 1 each. This results in the following bimatrix representation, where \( W \) stands for working, and \( S \) stands for shirking.

\[
\begin{array}{cc}
S & W \\
S & [0, 0] & [0, -c] \\
W & [-c, 0] & [1 - c, 1 - c] \\
\end{array}
\]
Find all the Nash equilibria of this game (both pure and mixed strategy equilibria). How does the mixed strategy equilibrium change as $c$ increases?

Let $\alpha_1(S) = p$ and $\alpha_2(S) = q$. The expected payoff of player 1 to actions $S$ and $W$ are

\[
\begin{align*}
    u_1(S, q) &= 0 \\
    u_1(W, q) &= (-c) \times q + (1 - c) \times (1 - q) \\
                &= 1 - q - c
\end{align*}
\]

Therefore, player 1 prefers to play $S$ if $0 > 1 - q - c$ or $q > 1 - c$, and prefers to play $S$ if $q < 1 - c$. If $q = 1 - c$ she is indifferent between $H$ and $D$. In other words,

\[
B_1(q) = \begin{cases}
    \{0\}, & \text{if } q < 1 - c \\
    [0, 1], & \text{if } q = 1 - c \\
    \{1\} & \text{if } q > 1 - c
\end{cases}
\]

where $B_1(q)$ is the best response of player 1 in terms of the choice of $p$.

Similarly, for player 2

\[
B_2(p) = \begin{cases}
    \{0\}, & \text{if } p < 1 - c \\
    [0, 1], & \text{if } p = 1 - c \\
    \{1\} & \text{if } p > 1 - c
\end{cases}
\]

Again, plot the best response correspondences to verify that this game has three Nash equilibria: two are pure strategy equilibria given by $p = q = 0$ (both players working), and $p = q = 1$ (both players shirking). The third one is a mixed strategy equilibrium given by $p = q = 1 - c$. As the cost of effort increases the probability with which players choose to shirk in the mixed strategy equilibrium decreases.