Economics of Information and Contracts
Signaling and Informed Principal Problem

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Signaling

Screening: agent is informed and principal uninformed
- Principal tries to reduce information rent of agent

Signaling: principal is the informed party
- Principal may signal private information through contract offer or action choice before contracting

Pioneering article:
- Spence, M. (1973)
  - Workers better informed about their productivity
  - They may use education as a signal
  - This works if education is costlier for lower productivity workers

Similar models in corporate finance:
  - Company owner may signal value of firm by retaining equity
  - New equity offers may reduce stock price

Signaling Games

- Two players
  - Player 1 is sender
    - has private information $\theta \in \Theta$
    - moves first by choosing an action $a_1 \in A_1$
  - Player 2 is receiver
    - observes $a_1$ but not $\theta$
    - moves by choosing an action $a_2 \in A_2$

- Nature’s probability distribution: $p \in \Delta(\Theta)$

- Payoffs: For all $a_1 \in A_1, a_2 \in A_2$ and $\theta \in \Theta$
  \[ u_i(a_1, a_2, \theta), \quad i = 1, 2 \]

- Strategies: $\beta_1(a_1 | \theta), \beta_2(a_2 | a_1)$
  \[ \beta_1 : \Theta \to \Delta(A_1) \]
  \[ \beta_2 : A_1 \to \Delta(A_2) \]

- Beliefs: $\mu(\theta | a_1)$

Expected Payoffs

- Since players may play randomized strategies and player 2 has incomplete information we use expected payoffs
- Given strategies and beliefs $(\beta_1, \beta_2, \mu)$
  - Expected payoff of player 1 of type $\theta$ if she plays $a_1$
    \[ U_1(a_1, \beta_2(a_2 | a_1), \theta) = \sum_{a_2} \beta_2(a_2 | a_1) u_1(a_1, a_2, \theta) \]
  - Expected payoff of player 2 after $a_1$ if he plays $a_2$
    \[ U_2(a_1, a_2, \mu) = \sum_{\theta} \mu(\theta | a_1) u_2(a_1, a_2, \theta) \]
Perfect Bayesian Equilibrium

Definition

A perfect Bayesian equilibrium is a collection of strategies and beliefs \((\beta_1, \beta_2, \mu)\) that satisfies

1. **Sequential Rationality**: Strategies maximize expected payoffs given beliefs
   
   1.1 \(\beta_1(a_1^*|\theta) > 0\) implies 
   
   \[ a_1^* \in \arg\max_{a_1} U_1(a_1, \beta_2(a_2|a_1), \theta) \]
   
   1.2 \(\beta_2(a_2^*|a_1) > 0\) implies 
   
   \[ a_2^* \in \arg\max_{a_2} U_2(a_1, a_2, \mu) \]

2. **Bayes’ Rule**: If there is a \(\theta\) such that \(\beta_1(a_1|\theta) > 0\)
   
   \[ \mu(\theta^*|a_1) = \frac{\beta_1(a_1|\theta^*)p(\theta^*)}{\sum_{\theta} \beta_1(a_1|\theta)p(\theta)} \]

An Example: Beer or Quiche

\[ \begin{array}{c|c|c|c|c|c|c|c} 
1 & 2 & 1 & 2 & 0 & 0 & 1 & 1 \\
1 & 2 & 1 & 2 & 0 & 0 & 1 & 1 \\
W & T & W & T & 0.1 & 0.9 & 0.1 & 0.9 \\
\end{array} \]

- \(\Theta = \{W, T\}\)
- \(A_1 = \{B, Q\}, A_2 = \{f, r\}\)
- \(p(W) = 0.1, p(T) = 0.9\)

Perfect Bayesian Equilibria of Beer or Quiche Game

Two classes of possible pure strategy equilibria

1. **Separating Equilibria**: different types choose different actions
2. **Pooling Equilibria**: both types choose the same action

Separating Equilibria

1. \(\beta_1(Q|T) = 1, \beta_1(B|W) = 1\)
   
   - Bayes rule (BR) \(\Rightarrow \mu(T|Q) = 1, \mu(W|B) = 1\)
   - Sequential rationality (SR) of 2 \(\Rightarrow \beta_2(r|Q) = 1, \beta_2(f|B) = 1\)
   - But SR of 1 \(\Rightarrow \beta_1(B|W) = 0\)
   - No such PBE
2. $\beta_1(B|T) = 1, \beta_1(Q|W) = 1$
   - Bayes rule (BR) $\Rightarrow \mu(T|B) = 1, \mu(W|Q) = 1$
   - Sequential rationality (SR) of 2 $\Rightarrow \beta_2(f|Q) = 1, \beta_2(r|B) = 1$
   - But SR of 1 $\Rightarrow \beta_1(Q|W) = 0$.
   - No such PBE

Pooling Equilibria
2. $\beta_1(Q|T) = \beta_1(Q|W) = 1$
   - Bayes rule (BR) $\Rightarrow \mu(W|Q) = 0.1, \mu(W|B) = \text{free}$
   - Sequential rationality (SR) of 2 $\Rightarrow \beta_2(r|Q) = 1, \beta_2(r|B)$
   - But SR of player 1 type $T$ $\Rightarrow \beta_2(f|B) = 1$
   - SR of 2 $\Rightarrow \mu(W|Q) \geq 1/2$

   The following is a class of PBE
   
   $\beta_1(B|T) = \beta_1(B|W) = 1, \beta_2(r|B) = 1, \beta_2(f|Q) = 1$
   
   $\mu(W|B) = 0.1, \mu(W|Q) \geq 1/2$

Intuitive Criterion
- Player 2’s beliefs after $B$ are not plausible
- Why would player 1 deviate and drink beer if he is type $W$
  - In equilibrium he is getting 0
  - Highest he can get by drinking beer is $-1$
- Player 1 of type $T$ has potentially something to gain
  - In equilibrium he gets $-1$
  - If he can convince player 2 that he is type $T$ he could get 0
- Player 2 should put zero probability on type $W$ after $B$
- But then he would play $r$ following $B$, upsetting the equilibrium
Intuitive Criterion

Cho and Kreps (1987) has formalized this intuition and called it intuitive criterion.

Take an equilibrium

- An action $a_2$ is undominated for player 2 after $a_1$ if there exists a belief under which $a_2$ is a best response to $a_1$.
- Best payoff of player 1 following $a_1$ is the maximum payoff that she can get when player 2 plays an undominated action.
- An action $a_1$ is equilibrium dominated for type $\theta$ if her best payoff following $a_1$ is strictly smaller than her equilibrium payoff.
- Player 2’s beliefs are reasonable after $a_1$ if it gives positive probability only to those types for whom $a_1$ is not equilibrium dominated.
- The equilibrium fails intuitive criterion if there exists a type and action $a_1$ for whom equilibrium payoff is smaller than the payoff to $a_1$, given that player 2 best responds to $a_1$ under reasonable beliefs.

Intuitive Criterion: Beer-Quiche

The pooling equilibrium in which both types eat quiche fails intuitive criterion.

$U_1(\beta, \mu|W) = 0$, $U_1(\beta, \mu|T) = -1$

- Note that
  $U_1(\beta, \mu|W) = 0 > -1 = \max_{a_2 \in BR(\theta, B)} U_1(\beta, a_2, \mu|W)$
  whereas
  $U_1(\beta, \mu|T) = -1 < 0 = \max_{a_2 \in BR(\theta, B)} U_1(\beta, a_2, \mu|T)$

- Therefore
  $J(\beta, \mu, B) = \{W\}$, $BR(\Theta \setminus J(\beta, \mu, B), B) = \{r\}$
  and
  $U_1(\beta, \mu|T) = -1 < 0 = \min_{a_2 \in BR(\Theta \setminus J(\beta, \mu, B), B)} U_1(\beta, a_2, \mu|T)$
An Application: Spence’s Model of Education

- A worker (player 1) has productivity (value produced per unit of time) equal to $L$ or $H$, with $H > L > 0$.
- The worker knows his productivity but the firm (player 2) only knows that the proportion of high productivity workers is $p > 0$.
- For any belief that the firm may hold about the worker’s productivity, the value of the worker to the firm is given by the expected productivity. We assume that the firm offers a wage $w$ that is equal to the expected productivity.
- We could model this by considering a labor market in which firms compete for the worker by offering wages.
- In such a model equilibrium wage would indeed be the expected productivity as long as firms have common beliefs.
- The worker chooses a level of education $e \geq 0$.
- The firm observes $e$ and makes a wage offer $w$.
- Payoff function of the worker is 

$$u_1(e, w, \theta) = w - \frac{e}{\theta}, \quad \theta = L, H$$

Separating Equilibria: $e_L \neq e_H$

- Bayes rule implies that $\mu(L|e_L) = 1$ and $\mu(H|e_H) = 1$.
- (1) implies $w(e_L) = L, w(e_H) = H$.
- SR of worker implies $e_L = 0$, since the worst that she can get by choosing $e_L = 0$ is $L$ (by equation (1)).
- SR implies that for all $e \geq 0$

$$L \geq w(e) - \frac{e}{L} \quad \text{and} \quad H - \frac{eH}{H} \geq w(e) - \frac{e}{H}$$

PBE of Spence’s Model of Education

- $e_\theta$: equilibrium education choice of worker with type $\theta$.
- $\mu(\theta|e)$: firm’s belief (probability) that productivity of the worker is $\theta$ if he chooses $e$ amount of education.
- Equilibrium wage schedule

$$w(e) = \mu(H|e)H + (1 - \mu(H|e))L$$

Separating Equilibria

- In particular we require

$$L \geq H - \frac{eH}{L} \quad \text{and} \quad H - \frac{eH}{H} \geq L$$

which is equivalent to

$$L(H - L) \leq eH \leq H(H - L)$$

- These are the incentive compatibility constraints.
- Bayes rule does not apply to beliefs after any $e \notin \{e_L, e_H\}$.
- The following belief and wage specification is one of many possible.

$$\mu(H|e) = \begin{cases} 1, & e \geq e_H \\ 0, & e < e_H \end{cases}, \quad w(e) = \begin{cases} H, & e \geq e_H \\ L, & e < e_H \end{cases}$$
Separating Equilibria

Proposition
An education profile \((e_H, e_L)\) is part of a pure strategy separating PBE iff \(e_L = 0\) and \(e_H \in [L(H - L), H(H - L)]\).

- These equilibria are Pareto ranked
- The best one is with \(e_H = L(H - L)\)
- It also is the only one that satisfies Intuitive Criterion

Pooling Equilibria: \(e_H = e_L = e^*\)

- Bayes rule implies \(\mu(H|e^*) = p\)
- Therefore, \(w(e^*) = pH + (1 - p)L \equiv E[\theta]\)
- Again we need, for any \(e \geq 0\)
  \[
  E[\theta] - \frac{e^*}{L} \geq w(e) - \frac{e}{L} \quad (2)
  \]
  \[
  E[\theta] - \frac{e^*}{H} \geq w(e) - \frac{e}{H} \quad (3)
  \]
- (2) and \(w(0) \geq L\) imply
  \[
  E[\theta] - \frac{e^*}{L} \geq L
  \]
  or
  \[
  e^* \leq pL(H - L)
  \]

Pooling Equilibria

- The following supports any such \(e^*\)
  \[
  \mu(H|e) = \begin{cases} p, & e \geq e^* \\ 0, & e < e^* \end{cases} \quad w(e) = \begin{cases} E[\theta], & e \geq e^* \\ L, & e < e^* \end{cases}
  \]

Proposition
An education profile \((e_H, e_L)\) is part of a pure strategy pooling PBE iff \(e_H = e_L \leq pL(H - L)\).

- Efficient pooling equilibrium has \(e_H = e_L = 0\)
Intuitive Criterion in Spence’s Model

**Proposition**
A pure strategy PBE satisfies intuitive criterion iff $e_L = 0, e_H = L(H - L)$.

**Proof**
We will first show that all pooling equilibria fail intuitive criterion. Let equilibrium education be $e^*$. $H > L$ implies that (verify) there exists an $e' > e^*$ such that

$$pH + (1 - p)L - \frac{e^*}{L} > H - \frac{e'}{L}$$
$$pH + (1 - p)L - \frac{e^*}{H} < H - \frac{e'}{H}$$

The left hand sides are equilibrium payoffs whereas the right hand sides are the maximum payoff that each type could get by playing $e'$ given that the firm plays a best response to some beliefs, in this case $\mu(H|e') = 1$.

**Proof (cont’d)**
Therefore, $e'$ is equilibrium dominated for $L$ and not for $H$. In our previous notation $J(\beta, \mu, e') = \{L\}$. Once we restrict the firm’s best response to beliefs $\mu(H|e') = 1$, the minimum payoff that type $H$ can get is bigger than the equilibrium payoff and hence the equilibrium fails intuitive criterion.

Now take a separating equilibrium in which $e_L = 0, e_H > L(H - L)$ and let $e' \in (L(H - L), e_H)$. We have

$$L > H - \frac{e'}{H}$$
$$H - \frac{e_H}{H} < H - \frac{e'}{H}$$

which implies that $e'$ is equilibrium dominated only for $L$. Given that $\mu(H|e') = 1$, playing $e'$ would bring at least $H - e'/H$ to type $H$, which is strictly better than the equilibrium payoff. Therefore, all separating equilibria in which $e_H > L(H - L)$ fail intuitive criterion.

**Proof (cont’d)**
Let us verify that separating equilibria in which $e_L = 0, e_H = L(H - L)$ satisfy intuitive criterion. If $e' > L(H - L)$, then

$$H - \frac{L(H - L)}{H} > H - \frac{e'}{H} > H - \frac{e'}{L}$$

and hence $J(\beta, \mu, e') = \{L, H\}$. But then the equilibrium payoff for type $\theta$ is at least as large as the minimum that he could get when the firm’s beliefs are not restricted: $L - e'/\theta$.

If, on the other hand, $e' < L(H - L)$, then

$$H - \frac{L(H - L)}{H} < H - \frac{e'}{H}$$

and hence $J(\beta, \mu, e') = \emptyset$. Again the equilibrium payoff for type $\theta$ is at least as large as the minimum that he could get when the firm’s beliefs are not restricted.
Intuitive Criterion

In both cases deviation to $e_d$ is profitable for the high type.

Figure 1: Pooling Equilibrium

Figure 2: Separating Equilibrium

Welfare Properties of Equilibria

There are multiple equilibria but only the best separating equilibrium satisfies the Intuitive Criterion Worker’s welfare under different scenarios:

1. Complete Information:
   - No education for anybody and $w_L = L, w_H = H$: best!

2. No Signaling: There is incomplete information but workers cannot signal
   - $w = pH + (1 - p)L$ and no education
   - low type is better off, high type is worse off compared to complete information
   - same outcome as efficient pooling equilibrium

3. Best Separating Equilibrium: $U_L = L$ and $U_H = H - \frac{L}{H}(H - L)$
   - Low type has the same payoff as complete information and is worse off compared to no-signaling case
   - High type is worse off compared to the complete information case
   - High type compared to no-signaling case
     - $p < 1 - L/H \Rightarrow$ better off
     - $p > 1 - L/H \Rightarrow$ worse off

Informed Principal Problem

Stage I Principal (worker) offers a contract: $(w_i, e_i)_{i=L,H}$

Stage II Agent (firm) accepts or rejects
   - If reject, both get zero
   - If accept, go to Stage III

Stage III Principal type $i$ chooses $e_i$ and gets $w_i$, $i = L, H$
   - Analyzed (in a general setting) by
     - Maskin, E. and J. Tirole (1992)
   - They showed that if $p < 1 - L/H$, then best separating equilibrium is the unique PBE
An Application: Adverse Selection and Signaling in Corporate Finance

A risk neutral entrepreneur has no funds to finance a project costing \( I \)

- Project yields \( R \) if successful and 0 if failure
- Project could be two types
  - High quality: probability of success is \( p_H \)
  - Low quality: probability of success is \( 0 < p_L < p_H \)

Entrepreneur observes the type of the project

- Lenders believe that the project is
  - High quality with probability \( q \)
  - Low quality with probability \( 1 - q \)
- Lenders are risk neutral, market for funds is competitive, and risk-free interest rate is zero
  - In equilibrium their expected payoff is zero
- Entrepreneur has limited liability: In case of failure she pays back 0

Two scenarios
1. \( p_H R > I > p_L R \): only high type is creditworthy
2. \( p_H R > p_L R > I \): both types are creditworthy

Perfect Information

- Entrepreneur observes quality and offers contract
- Feasible contracts: pay the lender \( D \) in case of success and 0 in case of failure
- Lenders accept only if their expected payoff is non-negative
- Suppose that lenders also observe the quality
- High type offers repayment \( D_H \) such that
  \[ p_H D_H = I \]
  and obtains expected payoff
  \[ p_H (R - D_H) = p_H R - I > 0 \]
- The best Low type can do is to offer \( D_L \) such that
  \[ p_L D_L = I \]
in which case her payoff is \( p_L R - I \)
- If \( p_L R > I \), she obtains financing, otherwise she does not

Asymmetric Information

- Lenders do not observe quality
- Separate Equilibria: \( D_L \neq D_H \)?
  - \( \mu(I|D_L) = 1, \mu(H|D_H) = 1 \)
  - If both types’ offers are accepted, zero profit condition implies
    \[ D_L = I/p_L > D_H = I/p_H \]
- Therefore, Low type mimics the High type
- There is no separating equilibrium in which Low type is denied lending either:
  \[ p_L (R - D_H) = p_L (R - I/p_H) > 0 \]
i.e., Low type again mimics
- There is no separating equilibrium

Asymmetric Information: Pooling Equilibria

- \( D_L = D_H = D \)
- Lenders’ expected payoff is \( pD - I \), where \( p = qp_H + (1 - q)p_L \)
- Two possibilities:
  - Lending Equilibrium
  - No lending Equilibrium
- Lending Equilibrium:
  - It must be that \( pR \geq I \)
    - Both types are creditworthy
    - or \( q \geq q^* \) where
      \[ (q^* p_H + (1 - q^*) p_L) R - I = 0 \]
  - \( I/p_L > D = I/p > I/p_H \)
    - High type is hurt; Low type benefits from asymmetric information
    - Lender may make losses on the Low type (cross-subsidization)
Asymmetric Information: Pooling Equilibria

- No Lending Equilibrium
- Implies $p_L R < I$: Otherwise
  - entrepreneur could offer $R - \varepsilon$, $\varepsilon > 0$ small
  - worst belief is $p_L \Rightarrow$ lender accepts it
- If $q < q^*$ where
  \[ (q^* p_H + (1 - q^*) p_L) R - I = 0 \]
- $\mu(H|D) = q$ for all $D$ and $D$ accepted iff $D \geq R$ is such an equilibrium
- market breakdown: even the worthy borrowers are denied credit

Asymmetric Information

- Note that we can write $p_R \geq I$ as
  \[ \left[ 1 - (1 - q) \frac{p_H - p_L}{p_H} \right] p_H R \geq I \]
  and define an index of adverse selection
  \[ \chi = (1 - q) \frac{p_H - p_L}{p_H} \]
- The Hight type’s pledgable income is discounted by the presence of Low types

Signaling: Equity Offering and Negative Stock Price Reaction

- It is well documented that stock prices decline upon announcement of new equity issue
  - Asquith and Mullins (1986)
  - Masulis and Korwar (1986)
- Model based on

Signaling: Equity Offering

- Entrepreneur already owns the project; without further investment yields $R$ with probabilities $p_L$ or $p_H$
- Entrepreneur knows the probability
- Investors put probability $q$ on $p_H$, $1 - q$ on $p_L$
- Since prob. of success is $p$
  - Assets in place undervalued for High type
  - overvalued for Low type
- Entrepreneur initially owns all shares
- Both types can increase prob. of success by $\tau$ by investing $I$
  \[ \tau R > I \]
  Investing in efficient for both types
- $I$ must be raised from investors by issuing new shares
  - Dilutes her ownership
  - Less costly if assets in place are overvalued
Equity Offering: Pooling Equilibrium

- Entrepreneur decides whether to announce equity offer \((E)\) or not \((N)\)
- Must offer a stake \(D\) such that 
  \[(p + \tau)D = I\]
- Without equity offering, High type gets \(p_H R\)
- Therefore, we need 
  \[(p_H + \tau)(R - D) \geq p_H R\] \hspace{1cm} (4)
  or 
  \[
  \tau R \geq \frac{p_H + \tau}{p + \tau} \]

\(\chi_T\) is the index of adverse selection
\[
\chi_T = \frac{(1 - q)(p_H - p_L)}{p_H + \tau}
\]

This is an efficient outcome
- Both types undertake investment
- No negative stock price reaction

Separating Equilibrium

- Low offers, High does not
- Beliefs of investors: \(\mu(L|E) = \mu(H|N) = 1\)
- Implies 
  \[(p_L + \tau)D = I\] \hspace{1cm} (5)
- High type must prefer not to offer 
  \[(p_H + \tau)(R - D) \leq p_H R\] \hspace{1cm} (6)
  or 
  \[
  \tau R \leq \frac{p_H + \tau}{p_L + \tau} \]
- Check that Low type prefers to offer \(D\)

Negative Stock Price Reaction

- Assume that investment opportunity is perfectly anticipated by the capital market
- Pre-announcement value of total shares: 
  \[V_0 = q[p_H R] + (1 - q)[(p_L + \tau)R - I]\]
- Post-announcement value 
  \[V_1 = (p_L + \tau)R - I\]
- (5) and (6) imply 
  \[p_H R \geq [(p_L + \tau)R - I] \frac{p_H + \tau}{p_L + \tau} > (p_L + \tau)R - I\]
  which implies \(V_1 < V_0\)
- This is negative stock price reaction
- Higher the volume of equity offer \(I\) and lower the value of project \(\tau\), higher the negative price reaction