Question 1. Kinetic energy and pressure of electron gas (Kittel 6.1-2)

(a) Consider a 3-d gas of N electrons at absolute zero. Since the model consists of free electrons, the internal energy is equal to total kinetic energy of the system. At 0 K the system is in its ground state, that is, the states above the state which has the Fermi energy $\epsilon_f$ are empty. Then the Fermi-Dirac distribution function (the probability that an orbital at energy $\epsilon$ will be occupied in an ideal electron gas in thermal equilibrium) is equal to a step function at zero temperature, namely 1 for $\epsilon \leq \epsilon_f$ and 0 for $\epsilon > \epsilon_f$ (See Kittel, Ch.6, Heat Capacity of the Electron Gas). The kinetic energy is then given by

$$ U = \int_0^{\epsilon_f} \, d\epsilon D(\epsilon), $$

where $D(\epsilon)$ is density of states. The number of states $N(K)$ for a wave vector $\vec{K}$ is given by

$$ N(K) = \frac{2 \pi K^3}{(2\pi L)^3} = \frac{K^3 V}{3\pi^2} \quad (2) $$

In order to find $N(\epsilon)$, the energy is $\epsilon_n = \frac{\hbar^2 K^2}{2m}$. Then, $N(\epsilon)$ and $D(\epsilon)$ are

$$ N(\epsilon) = \frac{V}{3\pi^2} \left( \frac{2m\epsilon}{\hbar^2} \right)^{3/2}, $$

$$ D(\epsilon) = \frac{dN}{d\epsilon} = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2}, \quad (3) $$

respectively. Using the above expression for density of states $D(\epsilon)$ in eq. (1) gives the kinetic energy for absolute zero, which is $U = \frac{3}{5} N \epsilon_f$.

(b) The pressure is given by $P = -\frac{dU}{dV}$. Using the expression for the kinetic energy in (a) and the fermi energy $\epsilon_f = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}$,

$$ U = \frac{3}{5} N \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}. \quad (4) $$

Ultimately, $P = -\frac{dU}{dV}$ is equal to $\frac{2U}{3V}$.

Question 2. Chemical Potential in 2D (Kittel 6.3)
In two dimensions, \( N(K) = \frac{2\pi K^2}{(2\pi)^2} \), where \( K^2 = \frac{2me}{\hbar^2} \), so \( N(\epsilon) = \frac{m\epsilon}{\pi\hbar^2} L^2 \). Density of states \( D(\epsilon) \) is given by

\[
D(\epsilon) = \frac{dN}{d\epsilon} = \frac{mL^2}{\pi\hbar^2}.
\] (5)

In the question, this result is given as a hint (note that you should multiply the expression in the hint with the area \( L^2 \), in order to reach the result above). For finite temperatures the number of electrons is equal to

\[
N = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon) = \frac{mL^2}{\pi\hbar^2} \int_0^\infty d\epsilon \frac{1}{e^{(\epsilon - \mu)/k_B T} + 1}.
\] (6)

It is easy to evaluate the integral when one multiplies both the numerator and denominator with \( e^{-(\epsilon - \mu)/k_B T} \), namely

\[
N = \frac{mL^2}{\pi\hbar^2} \int_0^\infty d\epsilon \frac{e^{-(\epsilon - \mu)/k_B T}}{1 + e^{-(\epsilon - \mu)/k_B T}}.
\] (7)

Defining \( x \equiv e^{-(\epsilon - \mu)/k_B T} + 1 \), the integral is rewritten as

\[
N = \frac{mL^2}{\pi\hbar^2} k_B T \int_1^{e^{\mu/k_B T} + 1} \frac{dx}{x}.
\] (8)

Then \( N \) is equal to

\[
N = \frac{mL^2}{\pi\hbar^2} k_B T \ln \left( e^{\mu/k_B T} + 1 \right).
\] (9)

The chemical potential \( \mu \) is obtained as

\[
\mu = k_B T \ln \left( \frac{e^{n\pi\hbar^2/mk_B T} - 1}{n} \right).
\] (10)

where \( n = \frac{N}{L^2} \).

**Question 3. Fermi gases in astrophysics (Kittel 6.4)**

(a) The Sun consists of approximately 75% Hydrogen and 25% Helium. The molar mass of He and H is nearly 4g and 1g, respectively. One H includes 1 electron, whereas one He has 2 electrons. So, for instance, in 7g of the Sun, there are \( 5N_a \) electrons, where \( N_a \) is Avogadro’s number. That is, in \( 2 \times 10^{33} \)g there are approximately \( 10^{57} \) electrons. Calculating the Fermi energy (in 3 dimensions),

\[
\epsilon_f = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3},
\] (11)
where $m_e$ is the mass of an electron, and $V = \frac{4}{3} \pi R^3$ ($R = 2 \times 10^9\text{cm}$), one finds that it is nearly equal to $3.5 \times 10^4\text{eV}$.

(b) In the relativistic limit, the Fermi energy is given by $\epsilon_f \simeq \hbar k_f c$. $k_f$ is proportional with $(\frac{N}{V})^{1/3}$. One finds that

$$\epsilon_f \simeq \hbar c \left(\frac{N}{V}\right)^{1/3}. \quad (12)$$

(c) "Pulsars are highly magnetized rotating neutron stars which emit a beam of detectable electromagnetic radiation in the form of radio waves" (Wikipedia). So their Fermi energy should be evaluated in the relativistic limit. From (a), $N_e \simeq 10^{57}$, and $V = \frac{4}{3} \pi \times 10^{12} \text{m}^3$. Setting these numerical values and the other constants into eq. (12), $\epsilon_f$ is found to be nearly $10^8\text{eV}$. 