KOÇ UNIVERSITY
College of Arts and Sciences
Department of Physics

Course: MATH503 Applied Mathematics
Credits: 3
Semester: Fall 2003
Instructor: Professor **Tekin Dereli**

Final Exam: 19 January 2004, 12.00-14.15

Question: 1 (20 points) Find the work done by the force $\vec{F} = xlny\vec{i} + 2ye^x\vec{j}$ when a body is taken along the boundary C of the rectangle $\{(x,y)|\, 0 \leq x \leq 2\,,\, 1 \leq y \leq 2\}$ in the clockwise sense.

Solution: Green's theorem

$$\int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} dA = \oint_{\partial S} \vec{F} \cdot d\vec{r}\,.$$

We have $\vec{\nabla} \times \vec{F} = (2ye^x - \frac{x}{y})\vec{k}$ so that

$$W = -\int_0^2 dx \int_1^2 dy (2ye^x - \frac{x}{y}) = 2ln2 - 3(e^2 - 1).$$

Note the over-all minus sign since the integration is to be done in the clockwise sense.

Question: 2 (20 points) What is the conic section represented by the quadratic form $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$. Determine its principal axes. Hint: Write the quadratic form as X^TAX where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and diagonalize A.

Solution: We identify A from

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and diagonalize it:

$$\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 0.$$

Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 16$. The corresponding eigenvectors

$$X_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad , \quad X_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

give the pricipal axes $L_1 : y = -\frac{1}{2}x$ and $L_2 : y = 2x$. In the oblique coordinates (x, y) determined by these lines, our quadratic reads

$$(x\ y) \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 16y^2 = 16.$$

This is the equation of an **ellipse**.

Question: 3 (20 points) Write down a general solution of the differential equation ($\omega \neq 1$)

$$y'' + \omega^2 y = sint.$$

Solution: Given the solution

$$y_1 = Acos\omega t + Bsin\omega t$$

of the homogeneous equation and a particular solution

$$y_2 = \frac{1}{(\omega^2 - 1)} sint$$

one can write down a general solution $y = y_1 + y_2$.

Question: 4 (20 points) Find and plot all the values of

$$(-8 + i8\sqrt{3})^{\frac{1}{4}}.$$

Solution: We have $-8 + i8\sqrt{3} = 16e^{i\frac{2\pi}{3}}$ so that its 4th roots are given by

$$z_n = 2e^{i(\frac{\pi}{6} + n\frac{\pi}{2})} \quad , n = 0, 1, 2, 3.$$

Explicitly we have $\sqrt{3} + i$, $-1 + i\sqrt{3}$, $-\sqrt{3} - i$, $1 - i\sqrt{3}$.

Question: 5 (20 points) Given the rational function

$$f(z) = \frac{z}{z^2+4}$$

i. find the zeroes and the poles of $f(z)$
ii. determine the residues at each pole
iii. integrate $f(z)$ counterclockwise along the simple closed path $C : |z| = 3$.

Solution: $f(z) = \frac{z}{(z-2i)(z+2i)}$ has one simple zero at $z_0 = 0$ and two simple poles at $z_1 = 2i$ and $z_2 = -2i$. Using partial fractions we can write

$$f(z) = \frac{1/2}{z-2i} + \frac{1/2}{z+2i}.$$

Therefore by inspection we read off the residues at each pole

$$Res_{z=2i} = 1/2 = Res_{z=-2i}.$$

Contour $C : |z| = 3$ encloses both the poles so that by the residue theorem

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{2} Res_{z=z_k} = 2\pi i(1/2 + 1/2) = 2\pi i.$$