Reformulations, Relaxations and Cutting Planes for Linear Generalized Disjunctive Programming

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Mixed Integer Program (MIP)
- Most common non-linear discrete/continuous optimization model.
- Purely equation-based.
- If all functions in MIP are linear $\rightarrow$ MILP (nonlinear $\rightarrow$ MINLP).

Disjunctive Programming (DP)
- Linear programming (LP) with disjunctive constraints.

Generalized Disjunctive Program (GDP)
- Combination of algebraic equations, disjunctions and logic propositions.
- Natural representation of engineering problems.

Mixed-Logic Linear Programming  
Hooker and Osorio (1999)

Constraint Programming  
Van Hentenryck (1989)
Motivation Hybrid Systems

Examples of work based on discrete/continuous optimization:

Benmporad and Morari (1999) MILP (HYSDEL software)
*Verification of Hybrid Systems via Mathematical Programming*

Strusberg and Panek (2002) GDP
*Control of Switched Hybrid Systems Based on Disjunctive Formulations*

Barton, Lee (2004) MINLP
*Design of process operations using hybrid dynamic optimization*
Goal:
To unify GDP with DP in order to develop MILP reformulations with improved relaxations for linear GDP

• To unify linear GDP with DP in order to develop
  - A hierarchy of LP relaxations for linear GDP
  - A family of disjunctive cutting planes for linear GDP

• Brief extension to Nonlinear and Bilinear GDPs
  - Approximation of convex hull and cutting plane algorithm
  - Tightening bounds of bilinear GDPs through basic steps
Linear Generalized Disjunctive Programming
LGDP Model


Objective function
\[ \text{Min } Z = \sum_{k \in K} c_k + d^T x \]

Common constraints
\[ s.t. \quad Bx \geq b \]

Disjunctive constraints
\[ \begin{bmatrix} Y_{jk} \\ A^{jk} x \geq a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K \]

Logic constraints
\[ \bigvee_{j \in J_k} Y_{jk} \quad k \in K \]
\[ \Omega(Y) = \text{True} \]
\[ x^L \leq x \leq x^U \]

Continuous variables
\[ Y_{jk} \in \{\text{True, False}\} \quad j \in J_k, k \in K \]

Boolean variables
\[ c_k \in \mathbb{R}^1 \quad k \in K \]

Logical OR operator
Process Network with fixed charges

GDP model

Min \( Z = c_1 + c_2 + c_3 + d^T x \)

s.t.

\[
\begin{align*}
    x_1 &= x_2 + x_4 \\
    x_6 &= x_3 + x_5 \\
    Y_{11} &= x_3 = p_1 x_2 \\
    Y_{12} &= x_5 = p_2 x_4 \\
    Y_{13} &= x_7 = p_3 x_6 \\
    c_1 &= \gamma_1 \\
    c_2 &= \gamma_2 \\
    c_3 &= \gamma_3 \\
Y_{21} &= x_3 = x_2 = 0 \\
Y_{22} &= x_5 = x_4 = 0 \\
Y_{23} &= x_7 = x_6 = 0 \\
\end{align*}
\]

\[
\begin{align*}
Y_{11} \lor Y_{21} \\
Y_{12} \lor Y_{22} \\
Y_{13} \lor Y_{23} \\
Y_{11} \lor Y_{12} \Rightarrow Y_{13} \\
Y_{13} \Rightarrow Y_{11} \lor Y_{12} \\
Y_{21} \lor Y_{22} \\
0 \leq x \leq x^U \\
Y_{11}, Y_{21}, Y_{12}, Y_{22}, Y_{13}, Y_{23} \in \{True, False\} \\
c_1, c_2, c_3 \in \mathbb{R}^I
\end{align*}
\]
LGDP to MILP Reformulations

Big-M Reformulation

\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + h^T x \\
\text{s.t. } & Bx \leq b \\
& A_{jk} x - a_{jk} \leq M_{jk} (1 - \lambda_{jk}) \quad j \in J_k, k \in K \\
& \sum_{j \in J_k} \lambda_{jk} = 1 \quad k \in K \\
& D\lambda \leq d \\
& x \in \mathbb{R}^n, \lambda_{jk} \in \{0,1\} \quad j \in J_k, k \in K
\end{align*}
\]

Note: \((M_{jk} = \max_{LB \leq x \leq UB} (A_{jk} x - a_{jk}), j \in J_k \text{ and } k \in K)\).

Relaxation: \(\lambda_{jk} \in \{0,1\}\) becomes \(0 \leq \lambda_{jk} \leq 1\) in (BM)
LGDP to MILP Reformulations
Convex Hull Reformulation

\[ \text{Min } Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + h^T x \]

s.t. \[ Bx \leq b \]
\[ A_{jk} \nu_{jk} - a_{jk} \lambda_{jk} \leq 0 \quad j \in J_k, \ k \in K \]
\[ x = \sum_{j \in J_k} \nu_{jk} \quad k \in K \quad (CH) \]
\[ 0 \leq \nu_{jk} \leq \lambda_{jk} U_{jk} \quad j \in J_k, \ k \in K \]
\[ \sum_{j \in J_k} \lambda_{jk} = 1 \quad k \in K \]
\[ D\lambda \leq d \]
\[ x \in \mathbb{R}^n, \ \nu_{jk} \in \mathbb{R}^n_+, \ \lambda_{jk} \in \{0,1\} \quad j \in J_k, \ k \in K \]

Relaxation: \( \lambda_{jk} \in \{0,1\} \) becomes \( 0 \leq \lambda_{jk} \leq 1 \) in \( (CH) \)
**Proposition:**

The Convex Hull of a set of disjunctions is the smallest convex set that includes that set of disjunctions. Furthermore, the projected relaxation of (CH) onto the space of (BM) is always as tight or tighter than that of (BM) \((Grossmann & Lee, 2002)\)

1. Tighter feasible region/lower bound \(\rightarrow\) less nodes \(\rightarrow\) decrease in computational solution time.
2. More variables and constraints \(\rightarrow\) more iterations \(\rightarrow\) increase in computational solution time.

Is Convex Hull best relaxation?
Disjunctive Programming

**Disjunction:** A set of constraints connected to one another through the logical OR operator \( \lor \)

**Conjunction:** A set of constraints connected to one another through the logical AND operator \( \land \)

Constraint set of a DP can be expressed in two equivalent extreme forms

- **Disjunctive Normal Form (DNF)**
  . A disjunction whose terms do not contain further disjunctions

\[
F = \left\{ x \in \mathbb{R}^n : \lor_{i \in Q} (A^i x \geq a^i) \right\}
\]

- **Conjunctive Normal Form (CNF)**
  . A conjunction whose terms do not contain further conjunctions

\[
F = \left\{ x \in \mathbb{R}^n : \Lambda x \geq \Lambda a, \lor_{h \in Q_j} (d^h x \geq d_0^h), \ j = 1, \ldots, t \right\}
\]
Linear Generalized Disjunctive Programming
LGDP Model


Objective function

Min \( Z = \sum_{k \in K} c_k + d^T x \)

s.t.

\( Bx \geq b \)

Common constraints

Disjunctive constraints

Logic constraints

Boolean variables

\[
\begin{align*}
\{ Y_{jk} \} & \quad j \in J_k, k \in K \\
\Omega(Y) & = True \\
x^L & \leq x \leq x^U \\
Y_{jk} & \in \{ \text{True}, \text{False} \} \\
c_k & \in \mathbb{R}^l \quad k \in K
\end{align*}
\]

How to deal with Boolean and logic constraints in Disjunctive Programming?
Reformulating LGDP into Disjunctive Programming Formulation


\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + d^T x \\
\text{s.t. } & Bx \geq b \\
& \bigvee_{j \in J_k} A^{jk} x \geq a^{jk} \\
& c_k = \gamma_{jk} \\
& \bigvee_{j \in J_k} Y_{jk} \\
& \Omega(Y) = \text{True} \\
& x^L \leq x \leq x^U \\
& Y_{jk} \in \{\text{True}, \text{False}\} \quad j \in J_k, k \in K \\
& c_k \in \mathbb{R}^1 \quad k \in K
\end{align*}
\]

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\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + d^T x \\
\text{s.t. } & Bx \geq b \\
& \bigvee_{j \in J_k} A^{jk} x \geq a^{jk} \\
& c_k = \gamma_{jk} \\
& \bigvee_{j \in J_k} \lambda_{jk} = 1 \\
& \sum_{j \in J_k} \lambda_{jk} = 1 \\
& H\lambda \geq h \\
& x^L \leq x \leq x^U \\
& 0 \leq \lambda_{jk} \leq 1 \quad j \in J_k, k \in K \\
& c_k \in \mathbb{R}^1 \quad k \in K
\end{align*}
\]

\text{LGDP} \quad \Rightarrow \text{Integrality } \lambda \text{ guaranteed}

\text{LDP} \quad \Rightarrow \text{Integrality } \lambda \text{ guaranteed}

Theorem. LGDP and LDP have equivalent solutions.
Equivalent Forms in DP Through Basic Steps

There are many forms between CNF and DNF that are equivalent

**Regular Form (RF):** form represented by intersection of unions of polyhedra

Thus the RF is:

\[ F = \bigcap_{t \in T} S_t \]

where for \( t \in T \), \( S_t = \bigcup_{i \in Q_t} P_i \), \( P_i \) a polyhedron, \( i \in Q_t \).

**Proposition 1 (Theorem 2.1 in Balas (1979)).** Let \( F \) be a disjunctive set in RF. Then \( F \) can be brought to DNF by \(|T| - 1\) recursive applications of the following basic steps, which preserve regularity:

For some \( r, s \in T, r \neq s \), bring \( S_r \cap S_s \) to DNF, by replacing it with:

\[ S_{rs} = \bigcup_{i \in Q_r, t \in Q_s} (P_i \cap P_t). \]
Illustrative Example: Basic Steps

\[ F = S_1 \cap S_2 \cap S_3 \]

\[ S_1 = (P_{11} \cup P_{21}) \quad S_2 = (P_{12} \cup P_{22}) \quad S_3 = (P_{13} \cup P_{23}) \]

Then \( F \) can be brought to DNF through 2 basic steps.

Apply Basic Step to:

\[ S_1 \cap S_2 = (P_{11} \cup P_{21}) \cap (P_{12} \cup P_{22}) \]

\[ S_{12} = (P_{11} \cap P_{12}) \cup (P_{11} \cap P_{22}) \cup (P_{21} \cap P_{12}) \cup (P_{21} \cap P_{22}) \]

We can then rewrite

\[ F = S_1 \cap S_2 \cap S_3 \quad \text{as} \quad F = S_{12} \cap S_3 \]

Apply Basic Step to:

\[ S_{12} \cap S_3 = ((P_{11} \cap P_{12}) \cup (P_{11} \cap P_{22}) \cup (P_{21} \cap P_{12}) \cup (P_{21} \cap P_{22})) \cap (P_{13} \cup P_{23}) \]

\[ S_{123} = \left( (P_{11} \cap P_{12} \cap P_{13}) \cup (P_{11} \cap P_{22} \cap P_{13}) \cup (P_{21} \cap P_{12} \cap P_{13}) \cup (P_{21} \cap P_{22} \cap P_{13}) \right) \]

\[ \quad \cup \left( (P_{11} \cap P_{12} \cap P_{23}) \cup (P_{11} \cap P_{22} \cap P_{23}) \cup (P_{21} \cap P_{12} \cap P_{23}) \cup (P_{21} \cap P_{22} \cap P_{23}) \right) \]

We can then rewrite

\[ F = S_{12} \cap S_3 \quad \text{as} \quad F = S_{123} \quad \text{which is its equivalent DNF} \]
Equivalent Forms for GDP

\[ \text{Min } Z = \sum_{k \in K} c_k + d^T x \]
\[ \text{s.t. } Bx \geq b \]
\[ \bigvee_{j \in J_k} A^{jk} x \geq a^{jk} \]
\[ c_k = \gamma^{jk} \]
\[ y_{jk} \in \{ \text{True}, \text{False} \} \]
\[ x^L \leq x \leq x^U \]
\[ c_k \in \mathbb{R}^I \]

**LGD P**

\[ \Omega(Y) = \text{True} \]
\[ x^L \leq x \leq x^U \]
\[ y_{jk} \in \{ \text{True}, \text{False} \} \]
\[ c_k \in \mathbb{R}^I \]

**LDP**

\[ \text{Min } Z = \sum_{k \in K} c_k + d^T x \]
\[ \text{s.t. } Bx \geq b \]
\[ \bigvee_{j \in J_k} A^{jk} x \geq a^{jk} \]
\[ c_k = \gamma^{jk} \]
\[ \sum_{j \in J_k} \lambda_{jk} = 1 \]
\[ H \lambda \geq h \]
\[ x^L \leq x \leq x^U \]
\[ 0 \leq \lambda_{jk} \leq 1 \]
\[ c_k \in \mathbb{R}^I \]

All possible equivalent forms for GDP, obtained through any number of basic steps, are represented by:

\[ F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\sum_{k \in K}|J_k|+|K|} : \bigcap_{i \in T} \bar{b}^i z \geq \bar{b}_0^i \bigcap_{k \in K} \bigcup_{j \in J_k} (\bar{A}^{jk} z \geq \bar{a}^{jk}) \bigcap_{n \in K} \bigcup_{m \in J_n} (\bar{A}^{mn} z \geq \bar{a}^{mn}) \right\} \]
Converting LDP to MIP reformulations

**Proposition 2** (Theorem 3.3 combined with Corollary 3.5 in Balas (1979)). Let
\[ F = \bigcup_{i \in Q} P_i, \quad P_i = \{ x \in \mathbb{R}^n : \tilde{A}^i x \geq \tilde{a}_0^i \}, \quad i \in Q, \]
where \( Q \) is an arbitrary set and each \((\tilde{A}^i, \tilde{a}_0^i)\) is an \( m_i \times (n+1) \) matrix such that every \( P_i \) is a **bounded non-empty polyhedron**.

Furthermore, let \( \zeta(Q) \) be the set of all those \( x \in \mathbb{R}^n \) such that there exist vectors \((v^i, y^i) \in \mathbb{R}^{n+1}, \quad i \in Q, \) satisfying
\[
\begin{align*}
  x - \sum_{i \in Q} v^i &= 0 \\
  \tilde{A}^i v^i - \tilde{a}_0^i y^i &\geq 0, \quad i \in Q \\
  y^i &\geq 0, \quad i \in Q \\
  \sum_{i \in Q} y^i &= 1, \quad i \in Q
\end{align*}
\]

Then \( \text{cl conv} \ F = \zeta(Q) \).

**Proposition 3** (Corollary 3.7 in Balas (1979)).

Let \( \zeta_I(Q) := \{ x \in \zeta(Q) : y_i \in \{0,1\}, \quad i \in Q \} \).

Then \( \zeta_I(Q) = F \).
Family of MIP Reformulations For GDP

\[ F = \left\{ z : (x, \lambda, c) \in \mathbb{R}^{n+\sum_{j \in J_k} |J_k|} : \cap_{i \in I} b_i^j z \geq b_0^j \cap_{j \in J_k} (\bar{A}^j z \geq \bar{a}^j) \cap_{m \in \mathcal{M}_n} (\bar{A}^{mn} z \geq \bar{a}^{mn}) \right\} \]

LDP’

General template for any MILP reformulation

\[ \text{Min } Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} y_{jk} + d^T x \]

s.t.

\[ b^i x \geq b_0^i \quad i \in I_B \]
\[ h^i y \geq h_0^i \quad i \in I_H \]
\[ x^L \leq x \leq x^U \quad i \in I_X \]
\[ 0 \leq y_{jk} \leq 1 \quad (j, k) \in L \]
\[ y_{jk} = \sum_{j \in J_k} \hat{u}_{jk} \quad (j, k) \in L_2 \cup K_{S_k} \cup I_{H_{S_k}}, \hat{k} \in \tilde{K} \]
\[ y_{jk} = \sum_{m \in \mathcal{M}_n} \hat{v}_{mn} \quad (j, k) \in L_3 \cup K_{S_n} \cup I_{H_{S_n}}, n \in \tilde{K} \]
\[ x = \sum_{j \in J_k} \hat{v}_{jk} \quad \hat{k} \in \tilde{K} \]
\[ x = \sum_{m \in \mathcal{M}_n} \hat{v}_{mn} \quad n \in \tilde{K} \]
\[ b^i \hat{v}_{jk} \geq b_0^i y_{jk} \quad i \in I_{B_2}, j \in J_k, \hat{k} \in \tilde{K} \]
\[ b^i \hat{v}_{mn} \geq b_0^i \tilde{y}_{mn} \quad i \in I_{B_3}, m \in J_n, \hat{n} \in \tilde{K} \]
\[ \sum_{j \in J_k} \hat{u}_{jk} \geq \hat{y}_{jk} \quad k \in K_{S_k}, j \in J_k, \hat{k} \in \tilde{K} \]
\[ \sum_{j \in J_k} \hat{u}_{jk} \geq \hat{y}_{mn} \quad k \in K_{S_n}, m \in J_n, n \in \tilde{K} \]
\[ h^i \hat{u}_{jk} \geq h_0^i \hat{y}_{jk} \quad i \in I_{H_k}, \bar{J} \in J_k, \hat{k} \in \tilde{K} \]
\[ h^i \hat{u}_{mn} \geq h_0^i \hat{y}_{mn} \quad i \in I_{H_n}, m \in J_n, n \in \tilde{K} \]

MIP’
Particular case: Convex Hull Reformulation of LGDP


\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} y_{jk} + d^T x \\
\text{s.t. } Bx &\geq b \\
x &= \sum_{j \in J_k} v_{jk} & k \in K \\
A^{jk} v_{jk} &\geq a^{jk} y_{jk} & j \in J_k, k \in K \\
x^L y_{jk} &\leq v_{jk} \leq x^U y_{jk} & j \in J_k, k \in K \\
\sum_{j \in J_k} y_{jk} &= 1 & k \in K \\
Hy &\geq h \\
y_{jk} &\in \{0,1\} & j \in J_k, k \in K
\end{align*}
\]

Disaggregated variables

While this MILP formulation has stronger relaxation than big-M, it is not strongest!!
A Hierarchy of Relaxations for DP

Hull Relaxation (Balas, 1985):

Let us take the following disjunctive set:

\[ F = \bigcap_{j \in T} S_j \]

Then the hull-relaxation corresponds to:

\[ h – rel\ F := \bigcap_{j \in T} clconv S_j. \]

Proposition 3 (Theorem 4.3 in Balas (1979)): For \( i = 0, 1, \ldots, t \), let \( F_i = \bigcap_{j \in T_i} S_j \) be a sequence of regular forms of a disjunctive set, such that

i) \( F_0 \) is in CNF, with \( F_0 = \bigcap_{j \in T_0} S_j \).

ii) \( F_t \) is in DNF;

iii) for \( i = 1, \ldots, t \), \( F_i \) is obtained from \( F_{i-1} \) by a basic step.

Then,

\[ P_0 = h – rel\ F_0 \supseteq h – rel\ F_1 \supseteq \cdots \supseteq h – rel\ F_t = clconv F_t. \quad (true\ convex\ hull) \]
A Hierarchy of Relaxations for GDP

Proposition 4. For \( i \in T \cup K \) let \( F_{\text{GDP}} \) be a sequence of regular forms of the disjunctive set:

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\sum_{k \in K} |J_k|} : \bigcap_{i \in T} b_i^i z \geq \bar{b}_0^i \bigcap_{k \in K} \bigcup_{j \in J_k} (\bar{A}^j z \geq \bar{a}^j) \bigcap \bigcup_{n \in K} \bigcup_{m \in J_n} (\hat{A}^m z \geq \hat{a}^m) \right\},
\]

such that

i) \( F_{\text{GDP}_0} \) corresponds to the disjunctive form:

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\sum_{k \in K} |J_k|} : \bigcap_{i \in T} b_i^i z \geq \bar{b}_0^i \bigcap_{k \in K} \bigcup_{j \in J_k} (\bar{A}^j z \geq \bar{a}^j) \right\};
\]

ii) \( F_{\text{GDP}_{|T|+|K|-1}} \) is in DNF;

iii) for \( i = 1, \ldots, t \), \( F_{\text{GDP}_i} \) is obtained from \( F_{\text{GDP}_{i-1}} \) by a basic step.

Then,

\[
h - \text{rel } F_{\text{GDP}_0} \supseteq h - \text{rel } F_{\text{GDP}_1} \supseteq \cdots \supseteq h - \text{rel } F_{\text{GDP}_{|T|+|K|-1}} = \text{clconv } F_{\text{GDP}_{|T|+|K|-1}} = \text{clconv } F_t. \text{(true convex hull)}
\]
Illustrative Example: Hierarchy of Relaxations

$\begin{align*}
  x_1 - x_2 + 0.5 &\geq 0 \\
  -x_1 - x_2 + 1 &\geq 0
\end{align*}$

\[
\begin{bmatrix}
  x_1 = 0 \\
  0 \leq x_2 \leq 1
\end{bmatrix} \lor \begin{bmatrix}
  x_1 = 1 \\
  0 \leq x_2 \leq 1
\end{bmatrix}
\]

Convex Hull of disjunction

Application of 2 Basic Steps

$\begin{align*}
  -x_1 - x_2 + 1 &\geq 0 \\
  x_1 = 0 \\
  0 \leq x_2 \leq 1
\end{align*}$

$\begin{align*}
  x_1 - x_2 + 0.5 &\geq 0 \\
  -x_1 - x_2 + 1 &\geq 0 \\
  x_1 = 1 \\
  0 \leq x_2 \leq 1
\end{align*}$

Convex Hull of disjunction

LP Relaxation

Tighter Relaxation!
Numerical Example: Strip-packing problem

Problem statement: *Hifi (1998)*

- Given a set of small rectangles with width $H_i$ and length $L_i$.
- Large rectangular strip of fixed width $W$ and unknown length $L$.
- Objective is to fit small rectangles onto strip without overlap and rotation while minimizing length $L$ of the strip.
GDP/DP Model for Strip-packing problem

Objective function
Minimize length

Disjunctive constraints
No overlap between rectangles

Bounds on variables

Minimize length

\[ \text{Minimize } \sum_{i \in N} x_i + L_i \]

Subject to:

- \[ x_i + L_i \leq x_j \]
- \[ x_i + L_j \leq x_i \]
- \[ y_i - H_i \geq y_j \]
- \[ y_j - H_j \geq y_i \]

forall \( i, j \in N, i < j \)

- \( x_i \leq UB_i - L_i \)
- \( H_i \leq y_i \leq W \)

\( l, x, y \in \mathbb{R}^n_+ \), \( Y^1_{ij}, Y^2_{ij}, Y^3_{ij}, Y^4_{ij} \in \{\text{True, False}\} \)

forall \( i, j \in N, i < j \)
# DP Model For 4 Rectangle Strip-packing Problem

Original CH formulation

<table>
<thead>
<tr>
<th>Min</th>
<th>lt</th>
</tr>
</thead>
<tbody>
<tr>
<td>st.</td>
<td>lt ≥ x_i + L_i ∀i ∈ N</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\lambda_{12} & = 1 \\
x_i + L_i & \leq x_2 \\
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\lambda_{13} & = 1 \\
x_i + L_i & \leq x_2 \\
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\lambda_{14} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{14} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{23} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{23} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{24} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{24} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{34} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{34} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{i} & \leq H_i \leq x_i \\
\forall i \in N \\
\lambda & \leq H_i \leq x_i \\
\forall i \in N \\
l_i, x_i, y_i & \in \mathbb{R}_+^+, 0 \leq \lambda_{ij}, \lambda_{ij}, \lambda_{ij}, \lambda_{ij} \leq 1 \\
\forall i, j \in N, i < j |
\end{align*}
\]

Optimum min \( L = 8 \)

Strengthened formulation

<table>
<thead>
<tr>
<th>Min</th>
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\lambda_{14} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{14} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{23} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{23} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{24} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{24} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{34} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{34} & = 1 \\
x_i + L_i & \leq x_2 \\
\lambda_{i} & \leq H_i \leq x_i \\
\forall i \in N \\
\lambda & \leq H_i \leq x_i \\
\forall i \in N \\
l_i, x_i, y_i & \in \mathbb{R}_+^+, 0 \leq \lambda_{ij}, \lambda_{ij}, \lambda_{ij}, \lambda_{ij} \leq 1 \\
\forall i, j \in N, i < j |
\end{align*}
\]

Optimum min \( L = 8 \)

Carnegie Mellon

102 variables (24 0-1), 143 constraints
LP relaxation = 4, No.nodes=45
Optimum min \( L = 8 \)

170 variables (60 0-1), 347 constraints
LP relaxation = 8, No.nodes=0
Optimum min \( L = 8 \)
25 Rectangle Problem Optimal solution = 31

Original CH
1,112 0-1 variables
4,940 cont vars
7,526 constraints
LP relaxation = 9

=>

Strengthened
1,112 0-1 variables
5,783 cont vars
8,232 constraints
LP relaxation = 27!

31 Rectangle Problem Optimal solution = 38

Original CH
2,256 0-1 variables
9,716 cont vars
14,911 constraints
LP relaxation = 10.64

=>

Strengthened
2,256 0-1 variables
11,452 cont vars
15,624 constraints
LP relaxation = 33!
Motivation for Cutting Plane Method

1. Tighter feasible region/lower bound $\rightarrow$ less nodes $\rightarrow$ decrease in computational solution time.
2. More variables and constraints $\rightarrow$ more iterations $\rightarrow$ increase in computational solution time.

Feasible region as tight as full space AND fewer variables and constraints
**DERIVATION OF CUTTING PLANES:**

**Dual perspective**

**Proposition 5.** The inequality \( az \geq a_0 \) is a consequence of

\[
F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\left|J_{i}\right|+\left|K\right|} : \bigcap_{i \in \hat{T}} \bigcap_{j \in J} \left( \bigcap_{k \in K \cap J} \left( A_{ik} \lambda \geq a_{ik} \right) \cup \left( A_{mn} z \geq \hat{a}_{mn} \right) \right) \right\},
\]

if and only if there exists a set of \( \hat{\theta}^j \geq 0, j \in J_k, k \in \hat{K} ; \hat{\theta}^{mn} \geq 0, m \in J_n, n \in \hat{K} ; \) and \( \Gamma^i, i \in \hat{T} \), satisfying

\[
\begin{align*}
\alpha & \geq \sum_{i \in \hat{T}} \Gamma^i \hat{b}^i + \sum_{k \in K} \hat{\theta}^{ik} \hat{A}^{ik} + \sum_{n \in K} \hat{\theta}^{mn} \hat{A}^{mn}, j, m \in J_n \\
\alpha_0 & \leq \sum_{i \in \hat{T}} \Gamma^i \hat{b}_0^i + \sum_{k \in K} \hat{\theta}^{ik} \hat{a}^{ik} + \sum_{n \in K} \hat{\theta}^{mn} \hat{a}^{mn}, j, m \in J_n.
\end{align*}
\]

Reverse Polar Cone
**Proposition 7.** \( \psi z \geq \alpha_0 \) with \( \alpha_0 \neq 0 \) is a facet of the \( h - \text{rel } F_{GDP} \), \( i = 0,1,\ldots,|\mathcal{T}| + |K| - 1 \) if and only if \( \alpha \neq 0 \) is a vertex of the polyhedron

\[
\left\{
\begin{array}{c}
\psi \in \mathbb{R}^{n + \sum_{k \in K} |J_k| + |K|} \\
\psi \geq \sum_{i \in J_n} \Gamma_i \hat{b}_i + \sum_{k \in K} \tilde{\theta}^k \hat{A}^k + \sum_{m \in J_n} \hat{\theta}^{nm} \hat{A}^{nm}, \ j \in J_k, m \in J_n \\
\alpha_0 \leq \sum_{i \in J_n} \Gamma_i \hat{b}_i + \sum_{k \in K} \tilde{\theta}^k \hat{A}^k + \sum_{m \in J_n} \hat{\theta}^{nm} \hat{A}^{nm}, \ j \in J_k, m \in J_n \\
\tilde{\theta}^k \geq 0, \ j \in J_k, k \in \hat{K} \\
\hat{\theta}^{nm} \geq 0, \ m \in J_n, n \in \hat{K} \\
\Gamma_i, \ i \in \mathcal{T}
\end{array}
\right.
\]
CUT GENERATION PROBLEM:
Dual perspective

Max $- \alpha z^{LP} + \alpha_0$

s.t.

$\alpha = \alpha^+ - \alpha^-$

$\sum_{i=1}^{n+\sum_{k \in K} |J_k|} (\alpha_i^+ - \alpha_i^-) \leq 1$

$\alpha \geq \sum_{i \in T} \Gamma^i b^i + \sum_{k \in K} \tilde{\alpha}^{jk} \tilde{A}^{jk} + \sum_{n \in \hat{K}} \hat{\alpha}^{mn} \hat{A}^{mn}, \ j \in J_k, m \in J_n$

$\alpha_0 \leq \sum_{i \in T} \Gamma^i b_0^i + \sum_{k \in \hat{K}} \tilde{\alpha}^{jk} \tilde{a}^{jk} + \sum_{n \in \hat{K}} \hat{\alpha}^{mn} \hat{a}^{mn}, \ j \in J_k, m \in J_n$

$\tilde{\alpha}^{jk} \geq 0, \ j \in J_k, k \in \hat{K}$

$\hat{\alpha}^{mn} \geq 0, \ m \in J_n, n \in \hat{K}$

$\Gamma^i, i \in \hat{T}$

$\alpha^+, \alpha^- \geq 0$

Normalization set
Balas, Ceria, Cornuejols (1993)

Reverse Polar Cone
CUT GENERATION PROBLEM: Primal perspective

\[ \begin{align*}
\text{Min} & \quad \eta \\
\eta & \geq z - z^{LP} \\
\eta & \geq z^{LP} - z \\
\bar{b}^i z & \geq \bar{b}_0^i \quad i \in \widehat{T} \\
z - \sum_{j \in J_k} \hat{\sigma}^{jk} & = 0 \quad k \in \widehat{K} \\
z - \sum_{m \in J_n} \hat{\sigma}^{mn} & = 0 \quad n \in \widehat{K} \\
\hat{A}^{jk} \hat{\sigma}^{jk} - \tilde{a}^{jk} y_{jk} & \geq 0 \quad j \in J_k, k \in \widehat{K} \\
\hat{A}^{mn} \hat{\sigma}^{mn} - \tilde{a}^{mn} \hat{y}_{mn} & \geq 0 \quad m \in J_n, n \in \widehat{K} \\
\sum_{j \in J_k} y_{jk} & = 1 \quad k \in \widehat{K} \\
\sum_{m \in J_n} \hat{y}_{mn} & = 1 \quad n \in \widehat{K} \\
y_{jk}, \hat{\sigma}^{jk} & \geq 0 \quad j \in J_k, k \in \widehat{K} \\
\hat{y}_{mn}, \hat{\sigma}^{mn} & \geq 0 \quad m \in J_n, n \in \widehat{K}
\end{align*} \]

Infinity Norm
Hull-relaxation
**Cut Generation Problem for Lee & Grossmann**

*(Separation Problem)*

Primal perspective separation problem *Saway, Grossmann (2006)*

\[
\begin{align*}
\text{Min } \phi(z) &= \| z - z^{bm} \| \\
\text{s.t.} \quad Bx &\leq b \\
A_{jk}v_{jk} &\leq a_{jk}v_{jk} \quad \forall j \in J_k, \forall k \in K \\
x &= \sum_{j \in J_k} v_{jk} \quad \forall k \in K \\
v_{jk} &\leq y_{jk}U_{jk} \quad \forall j \in J_k, \forall k \in K \\
\sum_{j \in J_k} y_{jk} &= 1 \quad \forall k \in K \\
Dy &\leq d \\
0 &\leq y_{jk} \leq 1 \quad \forall j \in J_k, \forall k \in K \\
x, v \in \mathbb{R}^n, z &= [x, y] \in \mathbb{R}_+^n \times \mathbb{R}^{\sum_{k \in K} |J_k|}
\end{align*}
\]

Note that $\phi(z) = \| z - z^{bm} \|_1$ or $\phi(z) = \| z - z^{bm} \|_2$ or $\phi(z) = \| z - z^{bm} \|_\infty$ can be used.

*Hull Relaxation for Lee & Grossmann*
Proposition 10: Let (FR-SEP) be the feasible region of the separation problem (SEP), and let (FRP-SEP) represent the projection of (FR-SEP) onto the z-space. Then, (FR-SEP) ⊆ (FR-BM), where (FR-BM) represents the feasible region of (BM). Furthermore, (FRP-SEP) is a convex set.

Proposition 11: Let $z^{bm}$ be the optimal solution of (BM) and $z^{sep}$ be an optimal solution to (SEP). If $z^{bm} \not\in$ (FRP-SEP), then $\exists \xi$ such that $\xi^T (z - z^{sep}) \geq 0$ is a valid linear inequality in $z$ that cuts away $z^{bm}$, and such that $\xi$ is a subgradient of $\phi(z)$ at $z^{sep}$.
# Cutting Plane Method
Derivation of Cutting Planes

**Propositions 12, 13, 14:**

1. Let \( \Phi (z) \equiv \| z - z^{bm} \|_2 \equiv (z - z^{bm})^T(z - z^{bm}) \). Then, 
   \[ \xi \equiv \nabla \Phi = (z - z^{bm}) \]

2. Let \( \Phi (z) \equiv \| z - z^{bm} \|_{\inf} \equiv \max_i |z_i - z_i^{bm}| \). Then, 
   \[ \xi \equiv (\mu^+ - \mu) \]
   
   \[ \begin{align*}
   \text{Min } u \\
   \text{s.t. } & u \geq z_i - z_i^{bm} \quad i \in I \quad \mu^+ \\
   & u \geq z_i^{bm} - z_i \quad i \in I \quad \mu \\
   \end{align*} \]

   Feasible region of (SEP)

3. Let \( \Phi (z) \equiv \| z - z^{bm} \|_1 \equiv \sum_i |z_i - z_i^{bm}| \). Then, 
   \[ \xi \equiv (\mu^+ - \mu) \]

   \[ \begin{align*}
   \text{Min } u \\
   \text{s.t. } & u_i \geq z_i - z_i^{bm} \quad i \in I \quad \mu^+ \\
   & u_i \geq z_i^{bm} - z_i \quad i \in I \quad \mu \\
   \end{align*} \]

   Feasible region of (SEP)
CUTTING PLANE METHOD

1. Solve relaxed Big-M MILP. This yields $z^{BM}$.
2. Solve separation problem.
   Feasible region corresponds to relaxed hull relaxation.
   Objective corresponds to finding point in hull relaxation closest to $z^{BM}$.
   This yields $z^{SEP}$.
3. Cutting plane is generated and added to relaxed big-M MILP.
4. Solve strengthened relaxed Big-M MILP. Go to 2.
NUMERICAL RESULTS
21-RECTANGLE STRIP PACKING PROBLEM

Table 3
Results for twenty one-rectangle strip-packing problem (CPLEX v. 8.1, default MIP options turned on)

<table>
<thead>
<tr>
<th></th>
<th>Relaxation</th>
<th>Optimal Solution</th>
<th>Gap (%)</th>
<th>Total Nodes in MIP</th>
<th>Solution Time for Cut Generation (sec)</th>
<th>*Total Solution Time (sec)</th>
<th>Number of Nodes per sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convex Hull</td>
<td>9.1786</td>
<td>---</td>
<td>---</td>
<td>968 652</td>
<td>0</td>
<td>&gt;10 800</td>
<td>89.69</td>
</tr>
<tr>
<td>Big-M</td>
<td>9</td>
<td>24</td>
<td>62.5</td>
<td>1 416 137</td>
<td>0</td>
<td>4 093.39</td>
<td>345.95</td>
</tr>
</tbody>
</table>

*Total solution time includes times for relaxed MIP(s) + LP(s) from separation problem + MIP.
## NUMERICAL RESULTS

### 21-RECTANGLE STRIP PACKING PROBLEM

Table 3

Results for twenty one-rectangle strip-packing problem (CPLEX v. 8.1, default MIP options turned on)

<table>
<thead>
<tr>
<th>Method</th>
<th>Relaxation</th>
<th>Optimal Solution</th>
<th>Gap (%)</th>
<th>Total Nodes in MIP</th>
<th>Solution Time for Cut Generation (sec)</th>
<th>*Total Solution Time (sec)</th>
<th>Number of Nodes per sec</th>
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<td>62.5</td>
<td>1 416 137</td>
<td>0</td>
<td>4 093.39</td>
<td>345.95</td>
</tr>
<tr>
<td>Big-M + 20 cuts</td>
<td>9.1786</td>
<td>24</td>
<td>61.75</td>
<td>306 029</td>
<td>3.74</td>
<td>917.79</td>
<td>334.80</td>
</tr>
<tr>
<td>Big-M + 40 cuts</td>
<td>9.1786</td>
<td>24</td>
<td>61.75</td>
<td>547 828</td>
<td>7.48</td>
<td>1 063.51</td>
<td>518.76</td>
</tr>
<tr>
<td>Big-M + 60 cuts</td>
<td>9.1786</td>
<td>24</td>
<td>61.75</td>
<td>28 611</td>
<td>11.22</td>
<td>79.44</td>
<td>419.32</td>
</tr>
<tr>
<td>Big-M + 62 cuts</td>
<td>9.1786</td>
<td>24</td>
<td>61.75</td>
<td>32 185</td>
<td>11.59</td>
<td>91.4</td>
<td>403.27</td>
</tr>
</tbody>
</table>

* Total solution time includes times for relaxed MIP(s) + LP(s) from separation problem + MIP
Non-linear Discrete/Continuous Optimization: GDP Model


\[
\begin{align*}
\text{Min } Z &= \sum_{k \in K} c_k + f(x) \\
\text{s.t. } r(x) &\leq 0 \\
\bigvee_{j \in J_k} Y_{jk} \quad &k \in K \\
\left[ \begin{array}{c}
Y_{jk} \\
g_{jk}(x) \leq 0 \\
c_k = \gamma_{jk}
\end{array} \right] \quad &k \in K \\
\bigvee_{j \in J_k} Y_{jk} \quad &k \in K \\
\Omega(Y) &= \text{True} \\
x^L \leq x \leq x^U \\
Y_{jk} \in \{\text{True, False}\} \quad &j \in J_k, k \in K \\
c_k \in \mathbb{R}_+ \quad &k \in K
\end{align*}
\]

Objective function
Common constraints
Disjunctive constraints
Logic constraints
Logical OR operator
Convex functions

Boolean variables

Carnegie Mellon
Cutting Plane Method

1. Solve relaxed (BM) problem. This yields \( z^{bm} = [x,y]^{bm} \).
2. Solve separation problem.
   Feasible region corresponds to relaxed (CH) problem.
   Objective corresponds to finding point in relaxed projected (CH) problem closest to \( z^{bm} \).
   This yields \( z^{sep} \).
3. Cutting plane is generated and added to relaxed (BM) problem.
Convex Hull Formulation

- **Consider Disjunction** $k \in K$

\[
\forall \ j \in J_k \left[ \begin{array}{c} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c = \gamma_{jk} \end{array} \right]
\]

- **Theorem:** Convex Hull of Disjunction $k$ \textit{(Lee, Grossmann, 2000)}
  - **Disaggregated variables** $v_j$
    \[
    \{(x, c) \mid x = \sum_{j \in J_k} v_j^k, \quad c = \sum_{j \in J} \lambda_{jk} \gamma_{jk}, \quad 0 \leq v_j^k \leq \lambda_{jk} U_{jk}, \quad j \in J_k \}
    \]
    \[
    \sum_{j \in J_k} \lambda_{jk} = 1, \quad 0 < \lambda_{jk} \leq 1,
    \]
    \[
    \lambda_{jk} g_{jk}(v_j^k / \lambda_{jk}) \leq 0, \quad j \in J_k \}
    \]
  - $\lambda_j$ - weights for linear combination

- Generalization of Stubbs and Mehrotra (1999)
Remarks

1. \( h(\nu, \lambda) = \lambda g(\nu / \lambda) \)

   If \( g(x) \) is a bounded convex function, \( h(\nu, \lambda) \) is a bounded convex function  \( \text{Hiriart-Urruty and Lemaréchal (1993)} \)

2. \( h(\nu,0) = 0 \) for bounded \( g(x) \)

3. For linear constraints convex hull reduces to result by \( \text{Balas (1985)} \)
Cutting Plane Method: Separation Problem

\[ \text{Min } \phi(z) = \| z - z^{bm} \| \quad (\text{SEP}) \]

\[ \begin{align*}
    \text{s.t.} & \quad r(x) \leq 0 \\
    & \quad \sum_{j \in J_k} y_{jk} g_{jk}(v_{jk} / y_{jk}) \leq 0 \\
    & \quad x = \sum_{j \in J_k} v_{jk} \\
    & \quad v_{jk} \leq y_{jk} U_{jk} \\
    & \quad \sum_{j \in J_k} y_{jk} = 1 \\
    & \quad Dv \leq d \\
    & \quad 0 \leq y_{jk} \leq 1 \\
    & \forall j \in J_k, \forall k \in K \\
\end{align*} \]

For \( \phi(z) = \| z - z^{bm} \|_2 \), we get \( (z^{sep} - z^{bm})(z - z^{sep}) \geq 0 \)

For \( \phi(z) = \| z - z^{bm} \|_1 \), we get \( (\mu^+ - \mu^-)(z - z^{sep}) \geq 0 \)

For \( \phi(z) = \| z - z^{bm} \|_{\infty} \), we get \( (\mu^+ - \mu^-)(z - z^{sep}) \geq 0 \)

How do we Implement this?

CONVEX NLP

(CH) Relaxed Feasible Region

Different Norms

Different Cuts
Computational Implementation of Separation Problem

Furman, Sawaya & Grossmann (2007)

Replace $y_{jk} g_{jk} (v_{jk} / y_{jk}) \leq 0$ by:

$0 \leq v_{jk} \leq U y_{jk}$

where

$$((1 - \varepsilon)v_{jk} + \varepsilon)(g_{jk} (v_{jk} /((1 - \varepsilon)v_{jk} + \varepsilon))) - \varepsilon g_{jk} (0)(1 - y_{jk}) \leq 0$$

1. The divisibility by 0 problem is avoided.

2. The new constraints are an exact approximation of the original constraints as $\varepsilon \to 0$.

3. The new constraints are an exact approximation of the original constraints at $v_{jk} = 0$ and at $v_{jk} = 1$ regardless of value of $\varepsilon$.

   if $y_{jk} = 0$,  $\Rightarrow (\varepsilon)(g_{jk} (0)) - \varepsilon g_{jk} (0) = 0 \leq 0$

   if $y_{jk} = 1$,  $\Rightarrow ((1)(g_{jk} (v_{jk} /1)) - \varepsilon g_{jk} (0)(0) = (1)g_{jk} (v_{jk} /1) \leq 0$

4. The LHS of the new constraints are convex.
Numerical Example
Design of Multi-product Batch Plant


- Design of batch plant with multiple units in parallel and intermediate storage tanks.
- Problem consists of determining volume of equipment, number of units in parallel and volume and location of intermediate storage tanks while minimizing investment cost.

![Diagram of multi-stage batch plant with questions on units in parallel per stage and storage tank size](image-url)
## Numerical Results
### Design of 10 Unit/Product Batch Plant

#### Problem Size

<table>
<thead>
<tr>
<th></th>
<th># of constraints</th>
<th># of variables</th>
<th># of discrete variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convex Hull</td>
<td>1296</td>
<td>637</td>
<td>89</td>
</tr>
<tr>
<td>Big-M</td>
<td>800</td>
<td>239</td>
<td>89</td>
</tr>
</tbody>
</table>

### Table 1: Results for design of 10 stage/product batch plant using traditional B&B (SBB)

<table>
<thead>
<tr>
<th></th>
<th>Relaxation</th>
<th>Optimal Solution</th>
<th>Gap (%)</th>
<th>Total Nodes in MINLP</th>
<th>Solution Time for Cut Generation (sec)</th>
<th>*Total Solution Time (sec)</th>
<th>Number of Nodes per sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convex Hull</td>
<td>650 401.14</td>
<td>729 948.49</td>
<td>10.9</td>
<td>5 359</td>
<td>0</td>
<td>711.76</td>
<td>7.53</td>
</tr>
<tr>
<td>Big-M</td>
<td>641 763.19</td>
<td>729 948.49</td>
<td>12.1</td>
<td>12 449</td>
<td>0</td>
<td>787.98</td>
<td>15.80</td>
</tr>
<tr>
<td>Big-M + 58 cuts</td>
<td>650 401.14</td>
<td>729 948.49</td>
<td>10.9</td>
<td>7 528</td>
<td>8.7</td>
<td>610.00</td>
<td>12.52</td>
</tr>
</tbody>
</table>

40% 23%
Global Optimization of Bilinear Generalized Disjunctive Programs

Juan Ruiz

Min \( Z = \sum_{k \in K} c_k \) \quad \rightarrow \quad \text{Objective Function}

s.t. \( g(x) \leq 0 \) \quad \rightarrow \quad \text{Global Constraints}

\( \bigvee_{i \in D_k} \begin{bmatrix} Y_{ik} \\ r_{ik}(x) \leq 0 \\ c_k = \gamma_{ik} \end{bmatrix} \) \quad \rightarrow \quad \text{Disjunctions}

\( k \in K \)

\( \Omega(Y) = \text{True} \) \quad \rightarrow \quad \text{Logic Propositions}

\( x \in \mathbb{R}^n, c_k \in \mathbb{R}, Y_{ik} \in \{\text{True, False}\} \quad i \in D_k, \quad k \in K \)

Bilinearities may lead to multiple local minima \( \rightarrow \) Global Optimization techniques are required

Relaxation of Bilinear terms using McCormick envelopes leads to a LGDP \( \rightarrow \) Improved relaxations for Linear GDP has recently been obtained (Sawaya & Grossmann, 2007)
Guidelines for applying basic steps in Bilinear GDP

- Replace bilinear terms in GDP by McCormick convex envelopes (LGDP)
- Apply basic steps between those disjunctions with at least one variable in common.
- The more variables in common two disjunctions have the more the tightening can be expected.
- If bilinearities are outside the disjunctions apply basic steps by introducing them in the disjunctions previous to the relaxation.
- If bilinearities are inside the disjunctions a smaller tightening effect is expected.
- A smaller increase in the size of the formulation is expected when basic steps are applied between improper disjunctions and proper disjunctions.
Case Study I: Water treatment network design

Process superstructure

Optimal structure

Generalized Disjunctive Program

Min \( Z = \sum_{k \in PU} CP_k \)

s.t.

\[ f_k^j = \sum_{i \in M_i} f_i^j \quad \forall j, k \in MU \]

\[ \sum_{i \in S_k} f_i^j = f_k^j \quad \forall j, k \in SU \]

\[ \sum_{i \in S_k} \zeta_i^k = 1 \quad k \in SU \]

\[ f_i^j = \zeta_i^k f_k^j \quad \forall j, i \in S_k \quad k \in SU \]

\[ F_k = \sum_j f_i^j, i \in OPU_k \quad \forall j \]

\[ CP_k = \partial_{ik} F_k \]

\[ 0 \leq \zeta_i^k \leq 1 \quad \forall j, k \]

\[ 0 \leq f_i^j, f_k^j \quad \forall i, j, k \]

\[ 0 \leq CP_k \quad \forall k \]

\[ YP_k^b \in \{\text{true, false}\} \quad \forall h \in D_k \quad \forall k \in PU \]
Case Study II: Pooling network design

Process superstructure

Stream i  Pool j  Product k

S1   P1  1
S2   P2  2
S3   P3  3
S4
S5   P4

Optimal structure

Stream i  Pool j  Product k

S1   P1  1
S2   P3  2
S5

N of cont. vars.: 76
N of disc. vars.: 9
N of bilinear terms: 24

Generalized Disjunctive Program

\[ \begin{align*}
\text{Min } Z &= \sum_{j \in J} CP_j + \sum_{i \in I} \sum_{j \in J} CST_i + \sum_{i \in I} \sum_{j \in J} \sum_{w \in W} c_{ijw} f_{ijw} - \sum_{k \in K} \sum_{j \in J} \sum_{w \in W} f_{jkw} \\
\text{s.t.} \quad & \sum_{i \in I} \sum_{j \in J} \sum_{w \in W} f_{ijw} = \sum_{k \in K} \sum_{j \in J} f_{jkw} \quad \forall j \in J \\
& \sum_{j \in J} \sum_{w \in W} f_{jkw} - S_k = 0 \quad \forall k \in K \\
& f_{ijw} = \lambda_{ijw} \sum_{w \in W} f_{ijw} \quad \forall i \in I, \forall j \in J, \forall w \in W \\
& \sum_{j \in J} \sum_{w \in W} f_{jkw} - Z_{kw} \sum_{j \in J} \sum_{w \in W} f_{jkw} = 0 \quad \forall k \in K, \forall w \in W \\
\end{align*} \]

\[ \begin{align*}
& YST_i \leq \sum_{j \in J} \sum_{w \in W} f_{ijw} \quad \forall i \in I \\
& \text{CST}_i = \alpha_i \quad \text{Min } Z^{*} = -4.640
\end{align*} \]

\[ \begin{align*}
& YP_j \leq \sum_{i \in I} \sum_{j \in J} f_{ijw} \quad \forall j \in J \\
& \sum_{k \in K} f_{jkw} = \sum_{i \in I} f_{ijw} \quad \forall w \in W \\
& f_{jkw} = \zeta_j^{k} \sum_{i \in I} f_{ijw} \quad \forall w \in W, k \in K \\
& \sum_{k \in K} \zeta_j^{k} = 1 \\
& \text{CST}_i, CP_j, YST_i, YP_j \in \{\text{true}, \text{false}\} \\
0 \leq \zeta_j^{k} \leq 1; 0 \leq f_{jkw}, f_{ijw} \leq f_{ijw}^{up} \\
0 \leq \text{CST}_i, CP_j, YST_i, YP_j \in \{\text{true}, \text{false}\}
\end{align*} \]
<table>
<thead>
<tr>
<th>Example</th>
<th>Initial Lower Bound</th>
<th>Global Optimization Technique using Lee &amp; Grossmann relaxation</th>
<th>Global Optimization Technique using proposed relaxation</th>
<th>Relative Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>400.66</td>
<td>499.86</td>
<td>99.7%</td>
<td>24.90%</td>
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<tr>
<td>Bound contraction</td>
<td></td>
<td></td>
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<td>99.7%</td>
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<tr>
<td>Nodes</td>
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<td>204</td>
<td></td>
<td>51%</td>
</tr>
<tr>
<td>Example 2</td>
<td>-5515</td>
<td>-5468</td>
<td>8%</td>
<td>0.90%</td>
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<tr>
<td>Bound contraction</td>
<td></td>
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<td>8%</td>
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<tr>
<td>Nodes</td>
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<td>683</td>
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<td>9%</td>
</tr>
</tbody>
</table>

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Conclusions

Unified GDP with Disjunctive Programming
- Developed DP equivalent formulation for GDP
- Developed a family of MIP reformulations for GDP
- Developed a hierarchy of relaxations for GDP

Developed framework for obtaining improved LP relaxations
- Demonstrated improved relaxations can be obtained compared to convex hull formulation Lee & Grossmann (2000)
- Numerical results have shown great improvement in lower bound for strip packing problem

Cutting Planes
- Showed equivalence dual and primal cut-generation problems.
- Developed a primal cut-and-branch algorithm where cutting planes were generated from the primal separation problem

Nonlinear GDPs
- Cutting planes can be readily extended
- Concept basic steps improves relaxation in bilinear GDPs

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