Limit theory for the domination number of random class cover catch digraphs

by

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Abstract

The CCCD problem is motivated by its applications in pattern classification. The domination number is an important measurement of the complexity of CCCD classifiers. Priebe *et al.* found the exact distribution of the domination number of CCCDs for the uniform distribution in one dimension. Under the same conditions, DeVinney and Wierman proved the Strong Law of Large Numbers (SLLN).

Based on DeVinney and Wierman's result, our research establishes the SLLN for general distributions in one dimension. In addition, we give an upper bound for the limiting value in the SLLN, which could lead to a statistical test for the equality of two distributions. After a lengthy calculation, we obtain the variance of the domination number in one dimension, and find the limiting variance. From this result, by using two limit theorems for negatively associated random variables, we prove the Central Limit Theorem (CLT) for the domination number in this one-dimensional case.

In two dimensions, we resort to "subadditive processes" to prove the Law of Large Numbers for the domination number. We first consider a Poissonized problem, then convert the SLLN in the Poisson case to the Weak Law of Large Numbers (WLLN) in the uniform distribution case. We finally generalize the WLLN to more general distributions, using the same idea in the one dimensional problem. At the end of this dissertation, we describe Monte Carlo simulations to empirically test the SLLN and CLT. The results support the limit theorems proved in this dissertation, and strongly suggest the CLT still holds in higher dimensions.

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Introduction

1.1 Mathematical Model

1.1.1 Class Cover Problem

The study of the class cover problem (CCP) was initiated by Cowen and Cannon [1], motivated by applications in statistical pattern classification [2]. Priebe *et al.* [3] introduced a general version of the CCP, described in the following way.

For a sample space Ω, a *dissimilarity function* $d : \Omega \times \Omega \to \mathbf{R}$ satisfies $d(\alpha, \beta) = d(\beta, \alpha) \geq d(\alpha, \alpha) = 0$ for all $\alpha, \beta \in \Omega$. Note that throughout this dissertation we will only consider the case where d is the Euclidean norm. Suppose there are two classes of Ω-elements, denoted as $\mathcal{X} \equiv \{x_i \in \Omega : i = 1, \cdots, n\}$ and $\mathcal{Y} \equiv \{y_j \in \Omega : j = 1, \cdots, m\}$. For each x_i, its *covering ball* is defined as follows.

Definition 1.1.1. $B(x_i) = \big\{\omega \in \Omega : d(\omega, x_i) < \min_j d(y_j, x_i)\big\}$.

A *class cover of* $\mathcal{X}$ is a subset of covering balls whose union contains all $x_i \in \mathcal{X}$. Suppose $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$ for any $\alpha, \beta \in \Omega$. If we assume each point in $\mathcal{X}$ is distinct from each point in $\mathcal{Y}$, then every covering ball $B(x_i)$ will be nonempty and contain x_i, hence the set consisting of all covering balls is a class cover. The CCP considered by Priebe *et al.* is to find a minimum cardinality class cover.

1.1.2 Class Cover Catch Digraph

The CCP can be converted to a graph theory problem, as follows.

Definition 1.1.2. *The class cover catch digraph (CCCD) induced by a CCP is the digraph* $D = (V, A)$ *with vertex set* $V = \{x_i : i = 1, \cdots, n\}$ *and arc set* $A = \big\{(x_i, x_j) : x_j \in B(x_i)\big\}$.

The definition above basically says that the CCCD induced by a CCP contains the directed edge from x_i to x_j if and only if x_j is included in the covering ball of x_i. An illustration of the construction of a CCCD is given in Figure 1.1. In this figure, the covering balls are drawn on the left as dashed circles, with the class X observations indicated by black dots and the class Y observations indicated by small circles; the induced CCCD is shown on the right.

A dominating set of a general digraph is defined as follows.

Definition 1.1.3. *The set* $S \subset V$ *is a dominating set of a digraph* $D = (V, A)$ *if and*

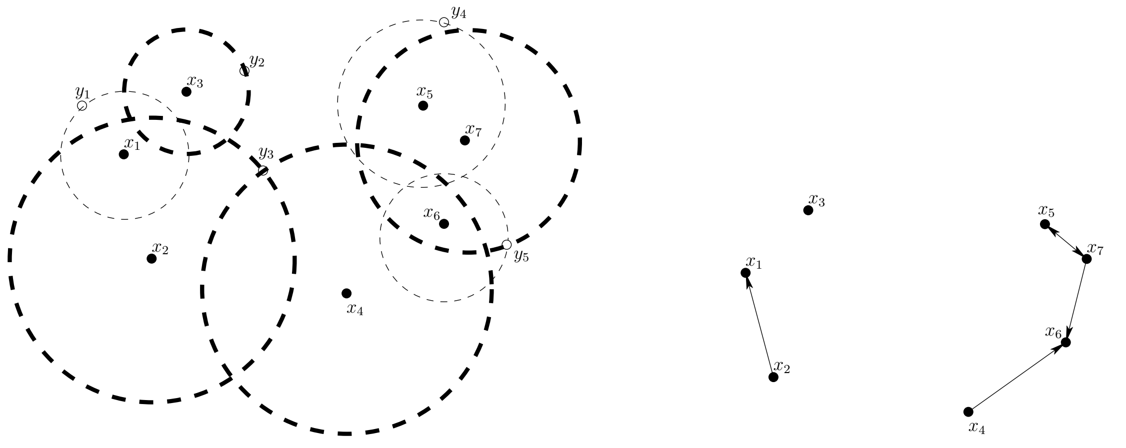

Figure 1.1: An illustration of the construction of a CCCD

only if for all $v \in V$*, either* $v \in S$ *or* $(s, v) \in A$ *for some* $s \in S$*.*

In other words, for any vertex v of a digraph, v is either contained in a dominating set, or there exists a directed edge from some vertex in the same dominating set to v. Recall that in a CCCD, there is a directed edge from x_i to x_j if and only if x_j is contained in the covering ball of x_i. Therefore, finding a minimum cardinality class cover of a CCP is equivalent to finding a minimum cardinality dominating set of the induced CCCD. We still use Figure 1.1 to illustrate this concept. In Figure 1.1, the darkened covering balls of x_2, x_3, x_4 and x_7 make up a minimum cardinality class cover, and $\{x_2, x_3, x_4, x_7\}$ is a minimum cardinality dominating set of the induced CCCD.

Note that the covering balls of x_2, x_3, x_4 and x_5 constitute another minimum cardinality class cover of the same CCP. Generally, there could be more than one solution to a CCP. Hence the minimum cardinality dominating set of a CCCD could be non-unique as well.

Finding a minimum cardinality dominating set in a general digraph is an NP-hard problem. However, this does not immediately imply that the CCP is NP-hard, since we have not characterized which digraphs are CCCDs. This topic is more thoroughly covered in DeVinney's dissertation [4].

1.1.3 Domination Number

Definition 1.1.4. *The domination number of a CCCD is the cardinality of the CCCD's minimum dominating set.*

Since Ore [5] first used the name "domination number" in 1962, there has been increasing interest in this topic because of its computational complexity and many applications in computer networks, social sciences and other fields. Haynes, Hedetniemi and Slater provide a comprehensive discussion of both the fundamentals [6] and advanced topics [7] of domination in graphs. In the CCCD setting, the domination number is the size of the minimum cardinality class cover, which in turn determines the complexity of the CCCD classifiers as shown in Section 1.2. This dissertation is devoted to the study of the domination number of CCCDs due to its theoretical importance in analyzing the CCCD classifiers.

1.1.4 Randomization

To study the problem from a statistical angle, randomness needs to be added to the Ω-valued points x_i and y_j. Specifically, x_i is replaced by a random variable X_i, and y_j is replaced by a random variable Y_j. We assume that the $X_i, i = 1, \cdots, n$ are independent of the $Y_j, j = 1, \cdots, m$, and all X_i's and Y_j's are distinct with probability one. After such randomization, all the previous definitions still apply. In particular, we let $\mathcal{X} \equiv \{X_i : i = 1, \cdots, n\}$ and $\mathcal{Y} \equiv \{Y_j : j = 1, \cdots, m\}$ be two sets of i.i.d. random variables taking values in Ω, with distribution functions F_X and F_Y, respectively. In addition, we denote the domination number by $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$, or simply $\Gamma_{n,m}$.

1.2 Applications in Pattern Classification

Pattern classification, "the assignment of a physical object or event to one of several prespecified categories" [8, page 2], has wide applications to various real world problems such as automated speech recognition, DNA sequence identification and fingerprint identification. For a thorough description of pattern classification, see the seminal texts by Duda *et al.* [9] and Lugosi *et al.* [10].

The abstract mathematical model of the pattern classification problem is formulated in the following way [2]. For simplicity, but without loss of generality, we suppose there are two classes of objects of interest, referred to as class X and class

Y, respectively. Assuming that the objects of both classes belong to a sample space Ω, to model the uncertainty about which class the objects belong to, we assume *prior* probabilities P_X and P_Y for these two classes $\big(\sum_{c\in\{X,Y\}} P_c = 1\big)$. We further assume that given the class, X or Y, the objects of that class are drawn according to the *class-conditional* distribution functions, $F_X(x)$ or $F_Y(y)$, respectively. A random pair $\big(c(\Psi), \Psi\big)$ is then generated in a two-step process: first we choose the random class label $c(\Psi) \in \{X, Y\}$ according to the prior probabilities; and then, based on the chosen class, we select Ψ according to the corresponding class-conditional distribution function.

In a classification problem, for an observation pair $\big(c(\psi), \psi\big)$ generated as above, only the data part ψ is given; the class label part $c(\psi)$ is unknown. Therefore, the goal of a *classifier* is to guess correctly whether $c(\psi)$ is X or Y. Given a training sample D_k of size k with known class labels

$$D_k = \Big\{ \big(c(\psi_1), \psi_1\big), \cdots, \big(c(\psi_k), \psi_k\big) \Big\},$$

then a classifier is a function $\hat{c}_k(\psi) = \hat{c}_k(\psi, D_k)$ that, based on the training data D_k, assigns a class label X or Y to any input point $\psi \in \Omega$. The performance of a classifier $\hat{c}$ is measured by the *probability of error*, or *misclassification rate*, given by

$$E\Big[P\big(\hat{c}_k(\Psi) \neq c(\Psi) \mid D_k\big)\Big].$$

The CCP has been actively studied recently because its solution can be directly used to generate classifiers competitive with other methods. The data point sets

$\mathcal{X} = \{X_i : i = 1, \cdots, n\}$ and $\mathcal{Y} = \{Y_j : j = 1, \cdots, m\}$ constitute the training data from classes X and Y, respectively. Thus, in the setting of classification, the CCP is simply a problem of selecting a small set of data points to be representative of a class. This set is chosen to be as small as possible, i.e., a minimal cardinality dominating set, to make the classifier less complex while keeping most of the relevant information. A simple CCCD classifier is constructed as follows: by switching the roles of X and Y, a pair of dual CCPs generates two solutions such as $\mathcal{B}_X = \big\{B(X_i) : i \in I\big\}, I \subset \{1, \cdots, n\}$, and $\mathcal{B}_Y = \big\{B(Y_j) : j \in J\big\}, J \subset \{1, \cdots, m\}$, respectively. If we define $\mathcal{C}_X = \big\{\omega \in \Omega : \omega \in B(X_i) \text{ s.t. } B(X_i) \in \mathcal{B}_X\big\}$, $\mathcal{C}_Y = \big\{\omega \in \Omega : \omega \in B(Y_j) \text{ s.t. } B(Y_j) \in \mathcal{B}_Y\big\}$, incorporating these two solutions gives a classifier $\hat{c}(\psi) : \Omega \to \{X, Y\}$ as follows:

$$\hat{c}(\psi) = \begin{cases} X & \psi \in \mathcal{C}_X \cap {\mathcal{C}_Y}^c, \\ Y & \psi \in \mathcal{C}_Y \cap {\mathcal{C}_X}^c, \\ \text{determined by further criteria} & \text{otherwise.} \end{cases}$$

More details about the CCP's application to classification are presented in Preibe *et al.* [11]. Note that the complexity of the classifier $\hat{c}(\psi)$ is determined by the sizes of $\mathcal{B}_X$ and $\mathcal{B}_Y$, i.e., the domination numbers $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ and $\Gamma_{m,n}(\mathcal{Y}, \mathcal{X})$. In other words, the domination number serves as a measure of efficiency in distinguishing the classes X and Y from each other.

1.3 Our Results

In this dissertation, we investigate the limit theory for the domination number of CCCDs. DeVinney and Wierman [12] have proved the the strong law of large numbers (SLLN) for the domination number generated by uniformly distributed data in one dimension (see Theorem 2.1.2). In Chapter 2, we extend their result to more general cases in which the class-conditional densities f_X and f_Y are bounded and continuous in any bounded interval.

To obtain the central limit theorem (CLT) for the domination number $\Gamma_{n,m}$ of CCCDs, we calculate the variance of $\Gamma_{n,m}$ (see Chapter 3). This calculation is conducted in one dimension for uniform class-conditional distributions. Under the same assumptions, in Chapter 4, we prove the CLT for the domination number in one dimension. An important tool used in this proof is *negative association*.

The work in two dimensions is more challenging because the exact distribution of $\Gamma_{n,m}$ is unavailable in any dimension higher than one. In Chapter 5, by using subadditive processes, we prove the SLLN for the domination number generated by Poisson points. Based on this result, we obtain the weak law of large numbers (WLLN) when the points are uniformly distributed in the unit square $[0,1]^2$. Then, applying the same technique used in Chapter 2, we generalize the WLLN to the case in which f_X and f_Y are positive, bounded and continuous.

In Chapter 6, we explore the empirical evidence for the CLT by Monte Carlo simulation methods. Although the CLT for the domination number in two dimensions is not obtained in this dissertation, we find that, as expected, the empirical distribution of the domination number in this situation is asymptotically normally distributed.

The techniques and ideas used in this research are not exclusive to the domination number. In Chapter 7, we present possibilities for applying our methods to other properties of CCCDs and suggest other research directions.

Chapter 2

SLLN for the Domination Number in One Dimension

Our first interest is to establish the SLLN for the domination number in one dimension. There have been several research results on the probabilistic properties of the domination number generated by uniform data. In Section 2.1, we introduce these previous results, which we will rely upon to prove the SLLN for continuous densities in later sections. The proof is first done for the case of piecewise constant densities in Section 2.2, and then extended to the case of continuous densities in Section 2.3. Finally, in Section 2.4, we discuss an upper bound for the limiting value in the SLLN, and its potential in building a statistical test for the equality of two distributions.

2.1 Previous Results

In one-dimensional space, we denote $Y_{(j)}$ as the jth order statistic of $Y_1, \cdots, Y_m$, and define $Y_{(0)} \equiv 0, Y_{(m+1)} \equiv 1$. The random variable $\alpha_{j,m}$ is defined as the minimum number of covering balls needed to cover the $N_{j,m}$ X-points located between $Y_{(j)}$ and $Y_{(j+1)}$. The random variables $\alpha_{0,m}$ and $\alpha_{m,m}$ are referred to as the external components, and $\alpha_{j,m}$ for $j = 1, \cdots, m-1$ are referred to as the internal components. It should be noted that $\Gamma_{n,m} = \sum_{j=0}^{m} \alpha_{j,m}$; thus, the original CCP is decomposed into $m+1$ sub-CCPs of finding the domination number $\alpha_{j,m}$ in the interval $[Y_{(j)}, Y_{(j+1)})$.

It is obvious that $\alpha_{j,m} = 0$ if and only if $N_{j,m} = 0$. It should also be noted that $\alpha_{j,m}$ is at most 2, because all X_i's in $[Y_{(j)}, Y_{(j+1)})$ are contained in the covering balls of the two X-points that are closest to the midpoint of this interval on the right and left. Since the domination number is a non-negative integer, $\alpha_{j,m}$ can only be $0, 1$ or 2. In particular, external components $\alpha_{0,m}$ and $\alpha_{m,m}$ can only be 0 or 1.

Through careful analysis, the probability of each of these values was determined exactly by Priebe, DeVinney and Marchette [3]. They found the conditional distribution of $\alpha_{j,m}$ given $N_{j,m}$ for the special case of $F_X = F_Y = U[0,1]$, where $U[0,1]$ is the uniform distribution on the interval $[0,1]$.

Theorem 2.1.1. *If $\Omega = \mathbf{R}$ and $F_X = F_Y = U[0,1]$, then the following are true:*

- *For $j \in \{0, 1, \cdots, m\}$, if $N_{j,m} = 0$ then $\alpha_{j,m} = 0$.*

- *For $j \in \{0, m\}$, if $N_{j,m} > 0$ then $\alpha_{j,m} = 1$.*

- *For* $j \in \{1, 2, \cdots, m-1\}$*, if* $N_{j,m} = n_{j,m} > 0$ *then*

$$\begin{aligned} P(\alpha_{j,m} = 1 \mid N_{j,m} = n_{j,m}) &= 1 - P(\alpha_{j,m} = 2 \mid N_{j,m} = n_{j,m}) \\ &= \frac{5}{9} + \frac{4}{9}\frac{1}{4^{n_{j,m}-1}}. \end{aligned}$$

Also, it should be noted that the internal components $\alpha_{j,m}, j = 1, \cdots, m-1$ are identically distributed, and the external components $\alpha_{j,m}, j = 0, m$ are also identically distributed.

The theorem above shows that for $j \in \{1, 2, \cdots, m-1\}$, given $N_{j,m} = n_{j,m} > 0$, the conditional probability of $\alpha_{j,m} = 2$ is an increasing function of $n_{j,m}$, meaning that for fixed m, each component $\alpha_{j,m}$ tends to become larger as the number of X-points increases.

Basing on Theorem 2.1.1, DeVinney and Wierman [12] proved the SLLN for uniform data, stated as follows:

Theorem 2.1.2. *If* $\Omega = \mathbf{R}, F_X = F_Y = U[0,1]$ *and* $m = \lfloor rn \rfloor, r \in (0, \infty)$*, then*

$$\lim_{n \to +\infty} \frac{\Gamma_{n,m}}{n} = g(r) \quad a.s.,$$

where

$$g(r) \equiv \frac{r(12r + 13)}{3(r+1)(4r+3)}.$$

The result above is equivalent to $\lim_{n \to +\infty} \frac{\Gamma_{n,m}}{m} = g(r)/r$ *a.s.* From the formula $g(r)$, it is apparent that $g(r)/r \to 0$ when $r \to \infty$, which is justified by the fact that

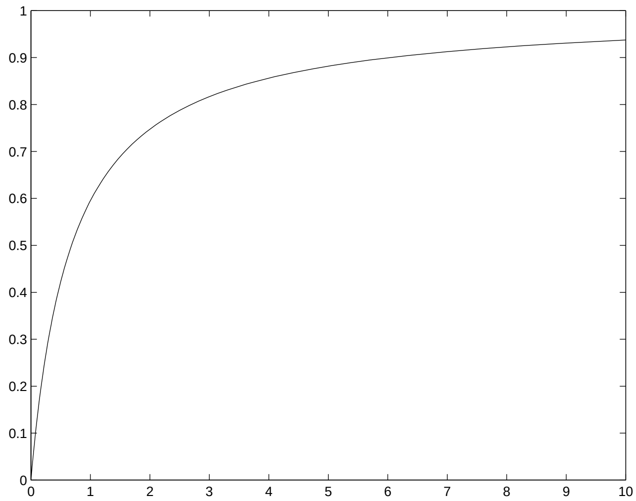

Figure 2.1: A graph of the limiting function $g(r)$, plotted using MATLAB.

asymptotically a typical interval between $Y_{(j)}$ and $Y_{(j+1)}$ almost certainly contains no X-point. Moreover, $g(r)/r \to \frac{13}{9}$ as $r \to 0$, which corresponds to the situation in which each interval between $Y_{(j)}$ and $Y_{(j+1)}$ contains a very large number of X-points. According to Theorem 1.1, the probability that $\alpha_{j,m} = 1$ is approximately $\frac{5}{9}$, while the probability that $\alpha_{j,m} = 2$ is approximately $\frac{4}{9}$, so $\frac{13}{9} = \frac{5}{9} \cdot 1 + \frac{4}{9} \cdot 2$ can be viewed simply as an asymptotic expectation value of $\alpha_{j,m}$.

In their proof [12], DeVinney and Wierman first treated the special case of $r = 1$. First, they constructed two Poisson processes, A and B, with a common rate $\lambda \in (0, \infty)$. A-points play the role of X-points, and B-points play the role of Y-points. There are a random number N_m of A-points before the $(m+1)$-st B-point. Conditional on the $(m+1)$-st arrival of the B process, the m B-points and N_m A-points before it are uniformly distributed. DeVinney and Wierman proved the complete convergence result for the domination number of the CCCD induced by these A-points and B-points. Hence the complete convergence result holds in the original setting with N_m X-points and m Y-points uniformly distributed on $[0, 1]$. Writing N_m as $m + G_m$, then for $0 < \epsilon \leq 1$, according to Chernoff's theorem,

$$\begin{aligned} P\left(\frac{|N_m - n|}{m} \geq \epsilon\right) &= P\left(\frac{|N_m - m|}{m} \geq \epsilon\right) \\ &= P\left(\frac{|G_m|}{m} \geq \epsilon\right) \\ &\leq C_1 e^{-\alpha_1(m\epsilon - 1)} + C_2 e^{-\alpha_2(m\epsilon - 1)} \end{aligned} \tag{2.1.1}$$

for all $m \geq 1$, where $\alpha_1, \alpha_2 > 0$ and C_1 and C_2 are constants. Thus, the difference between N_m and n is negligible in the limit. Based on the exponential bound above, complete convergence for the domination number in the case with N_m X-points is proved to be still true in the case with n X-points, and therefore almost sure convergence holds in the original setting.

For the $r \neq 1$ case, the proof can be extended by letting process A have rate $r\lambda$ and process B have rate λ.

In the following theorem, we weaken the condition of $m = \lfloor rn \rfloor$ in Theorem 2.1.2 to $m/n \to r$.

Theorem 2.1.3. *If* $\Omega = \mathbf{R}, F_X = F_Y = U[0, 1]$, *and* $m \equiv m(n)$ *with* $m/n \to r$ *as* $n \to \infty$ *where* $r \in (0, \infty)$, *then*

$$\lim_{n \to +\infty} \frac{\Gamma_{n,m}}{n} = g(r) \quad a.s,$$

where $g(r)$ *is the same as in Theorem 2.1.2.*

Proof. The whole proof is almost the same as that for Theorem 2.1.2, with the only difference given below. Given that $m/n \to r$, when n is sufficiently large, we have $\frac{|m-rn|}{m} \le \frac{\epsilon}{2}$. Hence, from Inequality (2.1.1), we get

$$\begin{aligned} P\left(\frac{\left|N_m - rn\right|}{m} \ge \epsilon\right) &= P\left(\frac{\left|m + G_m - rn\right|}{m} \ge \epsilon\right) \\ &\le P\left(\frac{|G_m|}{m} \ge \frac{\epsilon}{2}\right) \\ &\le C_1 e^{-\alpha_1(m\epsilon/2-1)} + C_2 e^{-\alpha_2(m\epsilon/2-1)}. \quad \square \end{aligned}$$

In the next two sections, we extend Theorem 2.1.3 to the general case in which the densities f_X and f_Y are bounded and continuous.

2.2 Piecewise Constant Densities

First, we consider a simpler situation in which f_X and f_Y are piecewise constant densities. Without loss of generality, the intervals of constancy for f_X and f_Y can be taken to be the same. Hence we suppose

$$\begin{aligned} f_X(x) &= \sum_{l=1}^{k} a_l I_{[c_{l-1},c_l)}(x), \\ f_Y(y) &= \sum_{l=1}^{k} b_l I_{[c_{l-1},c_l)}(y), \end{aligned}$$

where $a = c_0 < c_1 < \cdots < c_k = b$ and a_l, b_l are nonnegative. We define the following random variables:

$$\begin{aligned} N_l &= \left|\left\{X_i : X_i \in [c_{l-1}, c_l)\right\}\right|, \\ M_l &= \left|\left\{Y_j : Y_j \in [c_{l-1}, c_l)\right\}\right|. \end{aligned}$$

Lemma 2.2.1. *If $m/n \to r, r \in (0, \infty)$, then for $[c_{l-1}, c_l), l = 1, \cdots, k$, as $n \to \infty$,*

$$\begin{aligned} \frac{M_l}{m} &\to b_l(c_l - c_{l-1}) \quad a.s., \\ \frac{N_l}{n} &\to a_l(c_l - c_{l-1}) \quad a.s., \end{aligned}$$

and if $a_l \neq 0$, then

$$\frac{M_l}{N_l} \to r_l \quad a.s.,$$

where

$$r_l \equiv r \cdot \frac{f_Y(u)}{f_X(u)} = r\frac{b_l}{a_l} \quad \textit{for all } u \in [c_{l-1}, c_l).$$

Proof. Since $Y_j, j = 1, \cdots, m$, are i.i.d., the indicator random variables $I_{\{Y_j \in [c_{l-1}, c_l)\}}$ are also i.i.d. Therefore, by applying the standard SLLN, we get

$$\begin{aligned} \frac{M_l}{m} &= \frac{\left|\{Y_j : Y_j \in [c_{l-1}, c_l)\}\right|}{m} \\ &= \frac{\sum_{j=1}^{m} I_{\{Y_j \in [c_{l-1}, c_l)\}}}{m} \\ &\to E\left(I_{\{Y_j \in [c_{l-1}, c_l)\}}\right) \\ &= P\big(Y_j \in [c_{l-1}, c_l)\big) \\ &= b_l(c_l - c_{l-1}) \quad a.s. \end{aligned}$$

Similarly,

$$\frac{N_l}{n} \to a_l(c_l - c_{l-1}) \quad a.s.$$

Hence,

$$\begin{aligned}
\frac{M_l}{N_l} &= \frac{m \cdot \frac{M_l}{m}}{n \cdot \frac{N_l}{n}} \\
&\to r \cdot \frac{b_l(c_l - c_{l-1})}{a_l(c_l - c_{l-1})} \quad a.s. \quad \text{provided that } a_l \neq 0 \\
&= r_l. \quad \square
\end{aligned}$$

Dividing the original CCP into k sub-CCPs, each induced by $\mathcal{X}^l = \{X_i : X_i \in [c_{l-1}, c_l)\}$ and $\mathcal{Y}^l = \{Y_j : Y_j \in [c_{l-1}, c_l)\}$, $l = 1, \cdots, k$, we denote the cardinality of a minimum class cover of the l-th CCP by $\Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$. Since Lemma 2.2.1 shows that $M_l/N_l \to r_l$, from Theorem 2.1.3 it follows that

$$\frac{\Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)}{N_l} \to g(r_l) \quad a.s.$$

The points $c_l, l = 1, \cdots, k-1$, are referred to as "filter" points in that for each $l \in \{1, \cdots, k\}$, only X-points and Y-points in $[c_{l-1}, c_l)$ determine $\Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$. (Note that $c_0 = a$ and $c_k = b$ are fixed.) Recall that the domination number in one dimension is additive over intervals between Y-points. Specifically, we have $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) = \sum_{j=0}^{m} \alpha_{j,m}$, where each component $\alpha_{j,m}$ is determined by the X-points contained in $[Y_{(j)}, Y_{(j+1)})$. For any interval $[Y_{(j)}, Y_{(j+1)})$ containing no filter point, $\alpha_{j,m}$ must be a component of $\Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$ for the l such that $[Y_{(j)}, Y_{(j+1)}) \subset [c_{l-1}, c_l)$. However, if $[Y_{(j)}, Y_{(j+1)})$ contains one "filter" point c_l, then $\alpha_{j,m}$ is decomposed into the right external component of $\Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$ plus the left external component of $\Gamma_{n,m}(\mathcal{X}^{l+1}, \mathcal{Y}^{l+1})$. Finally, if $[Y_{(j)}, Y_{(j+1)})$ contains two or more "filter" points: $c_{l_1}, \cdots, c_{l_{T_j}}$ $(T_j \geq 2)$, then $\alpha_{j,m}$ is divided into the following $T_j + 1$ components: the right external component of

$\Gamma_{n,m}(\mathcal{X}^{l_1}, \mathcal{Y}^{l_1})$, plus $\Gamma_{n,m}(\mathcal{X}^{l_2}, \mathcal{Y}^{l_2}), \cdots, \Gamma_{n,m}(\mathcal{X}^{l_{T_j}}, \mathcal{Y}^{l_{T_j}})$, plus the left external component of $\Gamma_{n,m}(\mathcal{X}^{l_{T_j+1}}, \mathcal{Y}^{l_{T_j+1}})$. In summary, for any interval $[Y_{(j)}, Y_{(j+1)})$ containing no filter point, the corresponding component $\alpha_{j,m}$ of $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ is also a component of $\sum_{l=1}^{k} \Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$; for any interval $[Y_{(j)}, Y_{(j+1)})$ containing T_j filter points, the corresponding component $\alpha_{j,m}$ of $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ is decomposed into $T_j + 1$ components of $\sum_{l=1}^{k} \Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$. Furthermore, since any component mentioned above could only be $0, 1$ or 2 (see Theorem 2.1.1), the T_j "filter" points contained in a given interval $[Y_{(j)}, Y_{(j+1)})$ could contribute to the difference $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \sum_{l=1}^{k} \Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l)$ by at least $0 - 2 * (T_j + 1) = -2T_j - 2$ and at most $2 - 0 * (T_j + 1) = 2$. Supposing the set J consists of all j such that $[Y_{(j)}, Y_{(j+1)})$ contains at least one "filter" point, we have

$$\sum_{j \in J} (-2T_j - 2) \leq \Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \sum_{l=1}^{k} \Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l) \leq \sum_{j \in J} 2.$$

There are $k-1$ "filter" points, hence there are at most $k-1$ such intervals $[Y_{(j)}, Y_{(j+1)})$ that contain one or more "filter" points, thus $|J| \leq k - 1$. Therefore, from the inequality above we obtain

$$-2 \sum_{j \in J} T_j - 2(k-1) \leq \Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \sum_{l=1}^{k} \Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l) \leq 2(k-1).$$

By considering $\sum_{j \in J} T_j = k - 1$, the inequality above becomes

$$-4(k-1) \leq \Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \sum_{l=1}^{k} \Gamma_{n,m}(\mathcal{X}^l, \mathcal{Y}^l) \leq 2(k-1).$$

Since k is fixed,

$$\begin{aligned}\lim_{n\to\infty}\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})}{n} &= \lim_{n\to\infty}\sum_{l=0}^{k-1}\frac{\Gamma_{n,m}(\mathcal{X}^l,\mathcal{Y}^l)}{n}\\ &= \sum_{l=0}^{k-1}\lim_{n\to\infty}\frac{\Gamma_{n,m}(\mathcal{X}^l,\mathcal{Y}^l)}{N_l}\cdot\frac{N_l}{n}. \qquad (2.2.1)\end{aligned}$$

If $a_l \neq 0$, then by Lemma 2.1.3, $\frac{\Gamma_{n,m}(\mathcal{X}^l,\mathcal{Y}^l)}{N_l} \to g(r_l)$ $a.s.$, and by Lemma 2.2.1, $\frac{N_l}{n} \to a_l(c_l - c_{l-1})$ $a.s.$ Hence $\lim_{n\to\infty}\frac{\Gamma_{n,m}(\mathcal{X}^l,\mathcal{Y}^l)}{N_l}\cdot\frac{N_l}{n} = g(r_l)a_l(c_l - c_{l-1})$ $a.s.$ If instead $a_l = 0$, then, almost surely, there are no X-points in $[c_{l-1}, c_l)$, so $\Gamma_{n,m}(\mathcal{X}^l,\mathcal{Y}^l) = 0$ $a.s.$ Thus we still have $\lim_{n\to\infty}\frac{\Gamma_{n,m}(\mathcal{X}^l,\mathcal{Y}^l)}{n} = 0 = g(r_l)a_l(c_l - c_{l-1})$ $a.s.$ where $r_l = \infty$ and $g(\infty) \equiv \lim_{r\to\infty} g(r) = 0$. Therefore from Equation (2.2.1) we get

$$\lim_{n\to\infty}\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})}{n} = \sum_{l=1}^{k} g(r_l)a_l(c_l - c_{l-1}) \quad a.s.$$

Rewriting the expressions in the sum in the form of integrals generates

$$\begin{aligned}\lim_{n\to\infty}\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})}{n} &= \sum_{l=1}^{k}\int_{c_{l-1}}^{c_l} g\left(r\cdot\frac{f_Y(u)}{f_X(u)}\right) f_X(u)\,du\\ &= \int_a^b g\left(r\cdot\frac{f_Y(u)}{f_X(u)}\right) f_X(u)\,du \quad a.s.\end{aligned}$$

2.3 Continuous Densities

The formula obtained in the previous section is also valid when the densities f_X and f_Y are bounded and continuous. Specifically, we address the following main theorem in this chapter.

Theorem 2.3.1. *If $\Omega = \mathbf{R}$, the density functions f_X, f_Y are bounded and continuous on $[a,b]$, and $m/n \to r, r \in (0,\infty)$, then*

$$\lim_{n\to\infty} \frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})}{n} = \int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)}\right) \cdot f_X(u)du \quad a.s.,$$

where $g(r)$ is the same as in Theorem 2.1.2.

Proof. Since the density functions f_X, f_Y are bounded and continuous on $[a,b]$, f_X and f_Y are uniformly continuous. Thus for any $\epsilon > 0$, there exists a $\delta \equiv \delta(\epsilon) > 0$ such that for all x and y with $|x-y| < \delta$, $|f_X(x) - f_X(y)| \leq \frac{\epsilon}{4(b-a)}$ and $|f_Y(x) - f_Y(y)| \leq \frac{\epsilon}{4r(b-a)}$. Let $\Delta_l = [a+(l-1)\delta, a+l\delta) \cap [a,b]$ for $l \geq 1$. Define piecewise constant functions that approximate f_X and f_Y by

$$\begin{aligned}\bar{f}_X(x) &= \min\{f_X(u) : u \in \Delta_l\} \quad \text{for } x \in \Delta_l, \\ \bar{f}_Y(y) &= \min\{f_Y(u) : u \in \Delta_l\} \quad \text{for } y \in \Delta_l.\end{aligned}$$

Note that $\bar{f}_X$ and $\bar{f}_Y$ both depend on ϵ via δ; hence all functions and random variables derived from $\bar{f}_X$ and $\bar{f}_Y$ are also ϵ-dependent, but for simplicity we drop an explicit reference to ϵ throughout the proof.

Since $\bar{f}_X \leq f_X, \bar{f}_Y \leq f_Y$, it follows that $\int_a^b \bar{f}_X \leq 1$ and $\int_a^b \bar{f}_Y \leq 1$. Re-scaling $\bar{f}_X$ and $\bar{f}_Y$ gives density functions $\hat{f}_X$ and $\hat{f}_Y$, which approximate f_X and f_Y, respectively.

Our next step is to construct two classes of coupled random vectors: $\mathcal{X}$ vs. $\hat{\mathcal{X}}$, and $\mathcal{Y}$ vs. $\hat{\mathcal{Y}}$. Every component of the random vector $\mathcal{X}$ has density function f_X, whereas every component of $\hat{\mathcal{X}}$ has density function $\hat{f}_X$; and a similar property holds for $\mathcal{Y}$ and $\hat{\mathcal{Y}}$ as well. Now that we have introduced all the key notations, we first describe the overall structure of the proof before getting into the details. Recall that the ultimate goal is to prove that $\forall \eta > 0$, with probability 1, there exists an $N_\eta > 0$ such that, when $n > N_\eta$,

$$\left| \frac{\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})}{n} - \int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)} \right) f_X(u) du \right| \leq \eta. \tag{2.3.1}$$

Hence it suffices to prove that when $n > N_\eta$,

$$\left| \frac{\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})}{n} - \frac{\Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})}{n} \right| \leq \eta/3, \tag{2.3.2}$$

$$\left| \frac{\Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})}{n} - \int_a^b g\left(r \cdot \frac{\hat{f}_Y(u)}{\hat{f}_X(u)} \right) \hat{f}_X(u) du \right| \leq \eta/3, \tag{2.3.3}$$

and

$$\left| \int_a^b g\left(r \cdot \frac{\hat{f}_Y(u)}{\hat{f}_X(u)} \right) \hat{f}_X(u) du - \int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)} \right) f_X(u) du \right| \leq \eta/3. \tag{2.3.4}$$

We first consider Inequality (2.3.4). Note that the expressions inside the integral above are polynomials in the density functions $\hat{f}_X$ and $\hat{f}_Y$. Since as $\epsilon \to 0$, $\hat{f}_X(u) \to f_X(u)$ and $\hat{f}_Y(u) \to f_Y(u)$ for any $u \in [a, b]$, the Dominated Convergence Theorem gives

$$\int_a^b g\left(r \cdot \frac{\hat{f}_Y(u)}{\hat{f}_X(u)} \right) \cdot \hat{f}_X(u) du \to \int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)} \right) f_X(u) du \quad \text{as } \epsilon \to 0.$$

Thus, for any given η, there must exist an $\epsilon_\eta \leq \eta/3$ such that

$$\left| \int_a^b g\left(r \cdot \frac{\hat{f}_Y(u)}{\hat{f}_X(u)} \right) \cdot \hat{f}_X(u) du - \int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)} \right) f_X(u) du \right| \leq \eta/3,$$

where $\hat{f}_X$ and $\hat{f}_Y$ are constructed as described in the very beginning of the proof by choosing $\epsilon = \epsilon_\eta$. In the rest of this proof, we show that for the $\epsilon = \epsilon_\eta$, Inequalities (2.3.2) and (2.3.3) hold when n is sufficiently large.

We continue to describe the construction of $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$. First, consider i.i.d. random points $(X_{1i}, X_{2i}), 1 \le i \le n$, distributed uniformly over the region bounded by the x-axis, the line $x = a$, the line $x = b$, and the graph of f_X. Then,

$$P(s \le X_{1i} \le t) = \int_s^t f_X(u)du \quad \text{for all } a \le s \le t \le b,$$

so the marginal density function of X_{1i} is f_X. Similarly, construct i.i.d. random points $(Y_{1j}, Y_{2j}), 1 \le j \le m$, with Y_{1j}'s marginal density function being f_Y. Denote $\mathcal{X} = \{X_{1i} : i = 1, \cdots, n\}$ and $\mathcal{Y} = \{Y_{1j} : j = 1, \cdots, m\}$.

Next, let $(\bar{X}_{1i}, \bar{X}_{2i})$ and $(\bar{Y}_{1j}, \bar{Y}_{2j})$ be i.i.d. random points uniformly distributed over the regions under the graph of $\bar{f}_X$ and $\bar{f}_Y$ respectively. By the same argument as in the last paragraph, we can prove that the marginal density function of $\bar{X}_{1i}$ is $\hat{f}_X$, and the marginal density function of $\bar{Y}_{1j}$ is $\hat{f}_Y$.

Denote $\bar{R}_X$ as the region between the graphs of f_X and $\bar{f}_X$, and $\bar{R}_Y$ as the region between the graphs of f_Y and $\bar{f}_Y$. Finally, define

$$\begin{aligned}\left(\hat{X}_{1i}, \hat{X}_{2i}\right) &= \Big(X_{1i} I_{\{(X_{1i}, X_{2i}) \notin \bar{R}_X\}} + \bar{X}_{1i} I_{\{(X_{1i}, X_{2i}) \in \bar{R}_X\}}, \\ &\qquad X_{2i} I_{\{(X_{1i}, X_{2i}) \notin \bar{R}_X\}} + \bar{X}_{2i} I_{\{(X_{1i}, X_{2i}) \in \bar{R}_X\}}\Big)\end{aligned}$$

and

$$\begin{aligned}\left(\hat{Y}_{1j}, \hat{Y}_{2j}\right) &= \Big(Y_{1j} I_{\{(Y_{1j}, Y_{2j}) \notin \bar{R}_Y\}} + \bar{Y}_{1j} I_{\{(Y_{1j}, Y_{2j}) \in \bar{R}_Y\}}, \\ &\qquad Y_{2j} I_{\{(Y_{1j}, Y_{2j}) \notin \bar{R}_Y\}} + \bar{Y}_{2j} I_{\{(Y_{1j}, Y_{2j}) \in \bar{R}_Y\}}\Big).\end{aligned}$$

Here the idea is to set $\left(\hat{X}_{1i}, \hat{X}_{2i}\right) = \left(X_{1i}, X_{2i}\right)$ if $(X_{1i}, X_{2i}) \notin \bar{R}_X$, and $\left(\hat{X}_{1i}, \hat{X}_{2i}\right) =$

$\left(\bar{X}_{1i}, \bar{X}_{2i}\right)$ if $(X_{1i}, X_{2i}) \in \bar{R}_X$. The same idea applies for Y-points. Denote $\hat{\mathcal{X}} = \left\{\hat{X}_{1i} : i = 1, \cdots, n\right\}$ and $\hat{\mathcal{Y}} = \left\{\hat{Y}_{1j} : j = 1, \cdots, m\right\}$.

Lemma 2.3.1. *$\hat{X}_{1i}$ and $\hat{Y}_{1j}$ have piecewise constant density functions $\hat{f}_X$ and $\hat{f}_Y$, respectively.*

Proof. Any interval $[s, t] \subset [a, b]$ can be written as a disjoint union $\cup_{k=0}^{m}[s_k, t_k]$, where each $[s_k, t_k] \subseteq \Delta_k$ for distinct k. Denote $\hat{f}_X(\Delta_k) \equiv \hat{f}_X(x)$ for all $x \in \Delta_k$. If we can prove $P(s_k \leq \hat{X}_{1i} \leq t_k) = (t_k - s_k)\hat{f}_X(\Delta_k)$ for each k, then it follows that

$$P(s \leq \hat{X}_{1i} \leq t) \;=\; \sum_{k=0}^{m}(t_k - s_k)\hat{f}_X(\Delta_k),$$

hence $\hat{X}_{1i}$ has piecewise constant density functions $\hat{f}_X$. In fact, we know that for any k,

$$\begin{aligned} P\left(s_k \leq \hat{X}_{1i} \leq t_k\right) &= P\left(s_k \leq \hat{X}_{1i} \leq t_k \mid (X_{1i}, X_{2i}) \notin \bar{R}_X\right) P\left((X_{1i}, X_{2i}) \notin \bar{R}_X\right) \\ &\quad + P\left(s_k \leq \hat{X}_{1i} \leq t_k \mid (X_{1i}, X_{2i}) \in \bar{R}_X\right) P\left((X_{1i}, X_{2i}) \in \bar{R}_X\right). \end{aligned}$$

By recalling that $\left(\hat{X}_{1i}, \hat{X}_{2i}\right) = \left(X_{1i}, X_{2i}\right)$ if $(X_{1i}, X_{2i}) \notin \bar{R}_X$, and $\left(\hat{X}_{1i}, \hat{X}_{2i}\right) = \left(\bar{X}_{1i}, \bar{X}_{2i}\right)$ if $(X_{1i}, X_{2i}) \in \bar{R}_X$, the equation above becomes

$$\begin{aligned} P\left(s_k \leq \hat{X}_{1i} \leq t_k\right) &= P\left(s_k \leq X_{1i} \leq t_k \mid (X_{1i}, X_{2i}) \notin \bar{R}_X\right) P\left((X_{1i}, X_{2i}) \notin \bar{R}_X\right) \\ &\quad + P\left(s_k \leq \bar{X}_{1i} \leq t_k \mid (X_{1i}, X_{2i}) \in \bar{R}_X\right) P\left((X_{1i}, X_{2i}) \in \bar{R}_X\right). \end{aligned}$$

Observe that given $(X_{1i}, X_{2i}) \notin \bar{R}_X$, the random point (X_{1i}, X_{2i}) is bounded above by $\bar{f}_X$, hence the random variable X_{1i} has conditional density $\hat{f}_X$. Meanwhile, given

$(X_{1i}, X_{2i}) \in \bar{R}_X$, the conditional density of random variable $\bar{X}_{1i}$ is the same as its unconditional density $\hat{f}_X$, because $\bar{X}_{1i}$ is independent of X_{1i} and X_{2i}. Therefore,

$$\begin{aligned} P\left(s_k \leq \hat{X}_{1i} \leq t_k\right) &= (t_k - s_k)\hat{f}_X(\Delta_k)P\left((X_{1i}, X_{2i}) \notin \bar{R}_X\right) \\ &\quad + (t_k - s_k)\hat{f}_X(\Delta_k)P\left((X_{1i}, X_{2i}) \in \bar{R}_X\right) \\ &= (t_k - s_k)\hat{f}_X(\Delta_k). \end{aligned}$$

A similar result for $\hat{Y}_{1j}$ can be obtained by the same argument. □

Recall that the random variable $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ represents the size of a minimum class cover of $\mathcal{X} \equiv \{X_{1i}, i = 1, \cdots, n\}$ with respect to $\mathcal{Y} \equiv \{Y_{1j}, j = 1, \cdots, m\}$, and $\Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})$ represents the size of a minimum class cover of $\hat{\mathcal{X}} \equiv \{\hat{X}_{1i}, i = 1, \cdots, n\}$ with respect to $\hat{\mathcal{Y}} \equiv \{\hat{Y}_{1j}, j = 1, \cdots, m\}$. For any point $(X_{1i}, X_{2i}) \in \bar{R}_X$, we have set $\hat{X}_{1i} = \bar{X}_{1i}$, which is equivalent to replacing X_{1i} by $\bar{X}_{1i}$, hence the original domination number $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ changes. Note that deleting any X_{1i} can decrease the original domination number $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ by at most 1, while adding any $\bar{X}_{1i}$ can further decrease $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ by at most 1. Therefore, replacing X_{1i} by $\bar{X}_{1i}$ can contribute to the difference $\left|\Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})\right|$ by at most 2. Similarly, (Y_{1i}, Y_{2i}) in $\bar{R}_Y$ can also change the difference between $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ and $\Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})$ by at most 2. Thus,

$$\left|\Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})\right| \leq 2\left(\sum_{i=1}^{n} I_{\{(X_{1i}, X_{2i}) \in \bar{R}_X\}} + \sum_{i=1}^{m} I_{\{(Y_{1i}, Y_{2i}) \in \bar{R}_Y\}}\right).$$

Since the $I_{\{(X_{1i}, X_{2i}) \in \bar{R}_X\}}, i = 1, \cdots, n$ are i.i.d. random variables, applying the SLLN yields

$$\frac{\sum_{i=1}^{n} I_{\{(X_{1i},X_{2i})\in\bar{R}_X\}}}{n} \overset{n\to\infty}{\longrightarrow} E\left(I_{\{(X_{1i},X_{2i})\in\bar{R}_X\}}\right)$$

$$= P\left((X_{1i},X_{2i})\in\bar{R}_X\right)$$

$$\leq (b-a)\cdot\frac{\epsilon_\eta}{4(b-a)} = \frac{\epsilon_\eta}{4} \quad a.s., \tag{2.3.5}$$

and

$$\frac{\sum_{i=1}^{m} I_{\{(Y_{1i},Y_{2i})\in\bar{R}_Y\}}}{n} = \frac{m}{n}\cdot\frac{\sum_{i=1}^{m} I_{\{(Y_{1i},Y_{2i})\in\bar{R}_Y\}}}{m}$$

$$\overset{n\to\infty}{\longrightarrow} r\cdot E\left(I_{\{(Y_{1i},Y_{2i})\in\bar{R}_Y\}}\right)$$

$$= r\cdot P\left((Y_{1i},Y_{2i})\in\bar{R}_Y\right)$$

$$\leq r\cdot(b-a)\cdot\frac{\epsilon_\eta}{4r(b-a)} = \frac{\epsilon_\eta}{4} \quad a.s. \tag{2.3.6}$$

Recall that $\epsilon_\eta \leq \eta/3$. Consequently, with probability 1, there exists an $N'(\epsilon_\eta) > 0$ such that when $n > N'(\epsilon_\eta)$,

$$\left|\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y}) - \Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n}\right| \leq 2\cdot\left(\frac{\epsilon_\eta}{4}+\frac{\epsilon_\eta}{4}\right) = \epsilon_\eta \leq \eta/3,$$

which is exactly Inequality (2.3.2).

By the SLLN for piecewise constant densities, with probability 1, there exists an $N''(\epsilon_\eta) > 0$ such that when $n > N''(\epsilon_\eta)$,

$$\left|\frac{\Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n} - \int_a^b g\left(r\cdot\frac{\hat{f}_Y(u)}{\hat{f}_X(u)}\right)\hat{f}_X(u)\,du\right| \leq \epsilon_\eta \leq \eta/3,$$

which is exactly Inequality (2.3.3).

Therefore, both Inequalities (2.3.2) and (2.3.3) hold when $n > N_\eta \equiv \max\{N'(\epsilon_\eta), N''(\epsilon_\eta)\}$. Hence, Inequality (2.3.1) immediately follows as we have showed Inequality (2.3.4) holds when choosing $\epsilon = \epsilon_\eta$. Considering $\eta > 0$ is arbitrary, we conclude that

$$\lim_{n\to\infty} \frac{\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})}{n} = \int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)}\right) f_X(u)du \quad a.s. \quad \square$$

Remark 2.3.1. For simplicity, we assumed that the density functions f_X and f_Y are continuous and bounded. However, for some more general cases (e.g., when the densities are bounded but with only a finite number of discontinuities, when the densities have a finite number of vertical asymptotes, or when the densities are defined on unbounded intervals), our proof can apply as well by a slight modification. The key is to find appropriate piecewise constant functions $\bar{f}_X$ (bounded above by f_X) and $\bar{f}_X$ (bounded above by f_Y), where $\bar{f}_X$ and $\bar{f}_Y$ sufficiently approximate f_X and f_Y, respectively, so that Inequality (2.3.4), $P\big((X_{1i}, X_{2i}) \in \bar{R}_X\big) \leq \frac{\epsilon_\eta}{4}$ as in Inequality (2.3.5), and $P\big((Y_{1i}, Y_{2i}) \in \bar{R}_Y\big) \leq \frac{\epsilon_\eta}{4}$ as in Inequality (2.3.6) still hold.

2.4 An Upper Bound for the Limiting Value in the SLLN

We obtain the following upper bound for the limiting value in Theorem 2.3.1:

Corollary 2.4.1. *Under the same conditions as in Theorem 2.3.1, we have*

$$\int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)}\right) f_X(u)du \leq g(r),$$

where the equality holds if and only if $f_X = f_Y$ a.e.

Proof. From elementary calculus, $g(r)$ has a negative second derivative, so it is a strictly concave function. Let U denote the random variable that has density function f_X. Then, by applying Jensen's inequality on the function g, we get

$$\begin{aligned}\int_a^b g\left(r \cdot \frac{f_Y(u)}{f_X(u)}\right) f_X(u)du &\leq g\left(\int_a^b r \cdot \frac{f_Y(u)}{f_X(u)} f_X(u)du\right) \\ &= g(r).\end{aligned}$$

Since g is strictly concave, Jensen's theorem also tells us that the equality in the inequality above holds if and only if $r \cdot \frac{f_Y(U)}{f_X(U)} = E_X\left(r \cdot \frac{f_Y(U)}{f_X(U)}\right) = r$, i.e., $f_X = f_Y$ a.e. $\square$

The corollary above can be viewed in the following intuitive way. When class X and class Y both have the same distribution pattern, their objects tend to be interspersed. In this case, a larger domination number $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ would be needed to distinguish X from Y than when they have different distributions, because the domination number serves as a measure of efficiency in distinguishing the classes X and Y from each other.

We can apply Corollary 2.4.1 to build a distribution-free statistical test for the equality of two distributions, described as follows. Under the null hypothesis H_0 : $f_X = f_Y$ a.e., the limiting value of $\frac{\Gamma_{n,m}}{n}$ achieves the maximum value $g(r)$. Therefore, one form of statistical test would be to accept H_0 if $\frac{\Gamma_{n,m}}{n} > c_\alpha$, and reject H_0 otherwise, where the critical value c_α is determined by solving $P\left(\frac{\Gamma_{n,m}}{n} \leq c_\alpha \mid H_0\right) = \alpha$ for a given significance level α. Thus, in order to calculate the critical value, we need to know the distribution of the domination number $\Gamma_{n,m}$ under H_0. This issue is partly solved in Chapter 4 where we prove the CLT for $\Gamma_{n,m}$ when $F_X = F_Y = U[0,1]$. However, it remains as an open problem whether the CLT still holds when f_X and f_Y are any two equal densities. Although in this dissertation we haven't been able to establish the CLT for any equal densities other than the uniform densities, Monte Carlo simulation does suggest the CLT is still valid in this case (see Chapter 6).

Chapter 3

Variance of the Domination Number in One Dimension

As an important first step in proving the CLT for the domination number $\Gamma_{n,m}$, we need to compute its variance $Var(\Gamma_{n,m})$. By decomposing $\Gamma_{n,m}$ into internal and external components, we can write the variance as follows:

$$Var(\Gamma_{n,m}) \quad = \quad Var(\alpha_{0,m} + \sum_{j=1}^{m-1} \alpha_{j,m} + \alpha_{m,m}).$$

The formula above can be expressed as a sum of the variance and covariance of the components:

$$\begin{aligned} Var(\Gamma_{n,m}) &= 2Var(\alpha_{0,m}) + (m-1)Var(\alpha_{1,m}) \\ &\quad + 2\ Cov(\alpha_{0,m}, \alpha_{m,m}) + 2(m-1)Cov(\alpha_{0,m}, \alpha_{1,m}) + m(m-1)Cov(\alpha_{1,m}, \alpha_{2,m}), \end{aligned} \tag{3.0.1}$$

since the internal components are identically distributed and the external components are also identically distributed (refer to Lemma 3.2.2), and the covariance between any two components only depends on whether each component is an internal component or external component (refer forward to Lemma 3.2.3).

Thus, to obtain $Var(\Gamma_{n,m})$, we need to calculate the $Var(\alpha_{j,m}), j = 0, \cdots, m$, and $Cov(\alpha_{j_1,m}, \alpha_{j_2,m}), j_1, j_2 = 0, \cdots, m, j_1 \neq j_2$. Throughout this chapter we assume that $F_X = F_Y = U[0,1]$. Section 3.1 gives some facts about the distribution of $N_{j,m}$. In Section 3.2, we first calculate the conditional moments of $\alpha_{j,m}$ given $N_{j,m}$, and then we obtain the exact formula of $Var(\Gamma_{n,m})$ in terms of the expectations of some expressions involving $N_{j,m}$. In Section 3.3, based on the knowledge about the distribution of $N_{j,m}$, we further express $Var(\Gamma_{n,m})$ in terms of n and m. But such expression is too complicated to evaluate explicitly, so we settle for the limiting variance. In Section 3.4, we apply the dominated convergence theorem to determine the limiting value of $Var(\Gamma_{n,m})$ when $m/n \to r$ as $n \to \infty$.

3.1 Preliminary Distribution Facts

We let $Y_{(j)}$ denote the jth order statistic of $Y_1, \cdots, Y_m$, and define $Y_{(0)} \equiv 0, Y_{(m+1)} \equiv 1$. Let $L_{j,m} = Y_{(j+1)} - Y_{(j)}$ for $j = 0, \cdots, m$.

Proposition 3.1.1. *Given $L_{j,m} = l_{j,m} > 0, j = 0, \cdots, m$, the random vector $\{N_{j,m} : j = 0, \cdots, m\}$ is multinomially distributed with parameters $\{n, l_{j,m} : \sum_{j=0}^{m} l_{j,m} = 1\}$.*

Proof. Recall that $N_{j,m}$ is the number of X-points contained in the interval $[Y_{(j)}, Y_{(j+1)})$. Since the length of $[Y_{(j)}, Y_{(j+1)})$ is fixed as $l_{j,m}$ and the X-points are uniformly distributed on $[0,1]$, each X-point falls into $[Y_{(j)}, Y_{(j+1)})$ with probability $l_{j,m}$. Thus,

$$P(N_{0,m} = n_{0,m}, \cdots, N_{m,m} = n_{m,m}) = \binom{n}{n_{0,m}, \cdots, n_{m,m}} (l_{0,m})^{n_{0,m}} \cdots (l_{m,m})^{n_{m,m}},$$

where $\sum_{j=0}^{m} n_{j,m} = n$. □

Proposition 3.1.2. *For any different j_1, j_2, given $N_{j_1,m}$ and $N_{j_2,m}$, the corresponding components $\alpha_{j_1,m}$ and $\alpha_{j_2,m}$ are conditionally independent, and the conditional distribution of $\alpha_{j_1,m}$ is independent of $N_{j_2,m}$.*

Proof. When $N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}$, there are exactly $n_{j_1,m}$ X-points in $L_{j_1,m}$ and $n_{j_2,m}$ X-points in $L_{j_2,m}$, while all other X-points fall in $(L_{j_1,m} \cup L_{j_2,m})^c$. So the event $\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\}$ can be decomposed into a union of the following disjoint sub-events indexed by two disjoint subsets $\{s_1, \cdots, s_{n_{j_1,m}}\}$ and $\{t_1, \cdots, t_{n_{j_2,m}}\}$ of $\{1, \cdots, n\}$:

$$\begin{aligned} & A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}} \\ \equiv \; & \big\{X_i \in L_{j_1,m} \text{ for } i \in \{s_1, \cdots, s_{n_{j_1,m}}\}, X_i \in L_{j_2,m} \text{ for } i \in \{t_1, \cdots, t_{n_{j_2,m}}\}, \\ & X_i \in (L_{j_1,m} \cup L_{j_2,m})^c \text{ for } i \notin \{s_1, \cdots, s_{n_{j_1,m}}, t_1, \cdots, t_{n_{j_2,m}}\}\big\}. \end{aligned}$$

Given any sub-event $A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}$, $\alpha_{j_1,m}$ is a function of $X_{s_1}, \cdots, X_{s_{n_{j_1,m}}}$ and $\alpha_{j_2,m}$ is a function of $X_{t_1}, \cdots, X_{t_{n_{j_2,m}}}$, and these two groups of X-points are independent of each other, so $\alpha_{j_1,m}$ and $\alpha_{j_2,m}$ are also independent of each other.

Therefore, for any integer μ, ν,

$$
\begin{aligned}
&P\left(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right) \\
&= P\left(\alpha_{j_1,m} = \mu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right) \cdot P\left(\alpha_{j_2,m} = \nu \mid A_{\{s_1,\cdots,t_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right).
\end{aligned}
\tag{3.1.1}
$$

Recall that the event $\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\}$ is the union of the disjoint sub-events $A_{\{p_1,\cdots,p_{n_{j_1,m}}\},\{q_1,\cdots,q_{n_{j_2,m}}\}}$, hence

$$
\begin{aligned}
& P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= \;& \frac{P\Big(\alpha_{j_1,m} = \mu, \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)}{P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)} \\
= \;& \frac{\sum P\left(\alpha_{j_1,m} = \mu, A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right)}{P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)} \\
= \;& \frac{\sum P\left(\alpha_{j_1,m} = \mu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right) P\left(A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right)}{P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)}.
\end{aligned}
$$

Since the probability $P\left(\alpha_{j_1,m} = \mu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right)$ is the same for every $A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}$, we can factor out $P\left(\alpha_{j_1,m} = \mu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right)$ from the sum in the equation above. It follows that for any $A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}$,

$$
\begin{aligned}
& P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= \;& P\left(\alpha_{j_1,m} = \mu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right).
\end{aligned}
\tag{3.1.2}
$$

Similarly,

$$
\begin{aligned}
& P\Big(\alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= \;& P\Big(\alpha_{j_2,m} = \nu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\Big). \qquad (3.1.3)
\end{aligned}
$$

Plugging Equations (3.1.2) and (3.1.3) into Equation (3.1.1) yields

$$
\begin{aligned}
& P\left(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\right) \\
= \;& P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
& \cdot P\Big(\alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big). \qquad (3.1.4)
\end{aligned}
$$

Recall that the event $\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\}$ is the union of the disjoint sub-events $A_{\{p_1,\cdots,p_{n_{j_1,m}}\},\{q_1,\cdots,q_{n_{j_2,m}}\}}$, thus

$$
\begin{aligned}
& P\Big(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= & \frac{P\Big(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu, \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)}{P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)} \\
= & \frac{\sum P\Big(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu, A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\Big)}{P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)} \\
= & \frac{\sum P\Big(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu \mid A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\Big) P\Big(A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\Big)}{P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)}.
\end{aligned}
$$

By substituting Equation (3.1.4) into the formula above, we get

$$
\begin{aligned}
& P\Big(\alpha_{j_1,m} = \mu, \alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= & \sum P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
& \quad \cdot P\Big(\alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
& \quad \cdot P\Big(A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\Big) \Big/ P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= & P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
& \cdot P\Big(\alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
& \cdot \sum P\Big(A_{\{s_1,\cdots,s_{n_{j_1,m}}\},\{t_1,\cdots,t_{n_{j_2,m}}\}}\Big) \Big/ P\Big(\big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
= & P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) \\
& \cdot P\Big(\alpha_{j_2,m} = \nu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big).
\end{aligned}
$$

Hence $\alpha_{j_1,m}$ and $\alpha_{j_2,m}$ are independent given $\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\}$.

Furthermore, given the event $\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\}$, for any fixed subset $\{p_1, \cdots, p_{n_{j_1,m}}\}$ of $\{1, \cdots, n\}$, the event $A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}} \equiv \big\{X_i \in L_{j_1,m} \text{ for } i \in \{p_1, \cdots, p_{n_{j_1,m}}\}, X_i \in (L_{j_1,m})^c \text{ for } i \notin \{p_1, \cdots, p_{n_{j_1,m}}\}\big\}$ is independent of $n_{j_2,m}$. Thus,

$$
P\Big(A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}} \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big) = P\Big(A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}} \mid N_{j_1,m} = n_{j_1,m}\Big).
$$

Therefore, applying once again the technique of decomposing an event into the union of disjoint sub-events, we achieve the desired result as follows:

$$
\begin{aligned}
&P\Big(\alpha_{j_1,m} = \mu \mid \big\{N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\big\}\Big)\\
=&\sum P\left(\alpha_{j_1,m} = \mu \mid A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}}\right) \cdot P\left(A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}} \mid N_{j_1,m} = n_{j_1,m}, N_{j_2,m} = n_{j_2,m}\right)\\
=&\sum P\left(\alpha_{j_1,m} = \mu \mid A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}}\right) \cdot P\left(A'_{\{p_1,\cdots,p_{n_{j_1,m}}\}} \mid N_{j_1,m} = n_{j_1,m}\right)\\
=&P\big(\alpha_{j_1,m} = \mu \mid N_{j_1,m} = n_{j_1,m}\big). \quad \square
\end{aligned}
$$

Proposition 3.1.3. *Supposing that $Y_j, j = 1, \cdots, m$ are uniformly distributed on $[0, 1]$, we let $Y_{(j)}, j = 1, \cdots, m$ denote the order statistics of $Y_1, \cdots, Y_m$, and define $Y_{(0)} \equiv 0, Y_{(m+1)} \equiv 1$. If we define $L_j = Y_{(j+1)} - Y_{(j)}, j = 0, \cdots, m$, then the density function of L_j is*

$$f_{L_j}(l_j) = m(1 - l_j)^{m-1},$$

and the joint density function of L_i and L_j is

$$f_{L_{j_1},L_{j_2}}(l_{j_1}, l_{j_2}) = m(m-1)(1 - l_{j_1} - l_{j_2})^{m-2}.$$

Proof. We know that the joint density function of order statistics $Y_{(1)}, \ldots, Y_{(m)}$ is

$$m! \prod_{j=1}^{m} f_Y(y_j),$$

where f_Y is the common marginal distribution of $Y_j, j = 1, \cdots, m$. Since $f_Y(y_j) = I_{y_j \in [0,1]}$, we have

$$f_{Y_{(1)},\ldots,Y_{(m)}}(y_{(1)}, \ldots, y_{(m)}) = \begin{cases} m! & 0 = y_{(0)} \le y_{(1)} \le \cdots \le y_{(m)} \le y_{(m+1)} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the marginal density function of $(Y_{(j)}, Y_{(j+1)})$ is

$$\begin{aligned} f_{Y_{(j)},Y_{(j+1)}}(y_{(j)}, y_{(j+1)}) &= \underset{0\le y_{(j)}<y_{(j+1)}\le 1}{\int\cdots\int} f(y_{(1)},\ldots,y_{(m)})\,dy_{(1)}\ldots dy_{(j-1)}\,dy_{(j+2)}\ldots dy_{(m)} \\ &= \frac{m!}{(j-1)!(m-j-1)!} y_{(j)}^{j-1}(1-y_{(j+1)})^{m-j-1}. \end{aligned}$$

Applying the distribution theory of transformations of random vectors using $z = y_{(j)}$ and $z + l_j = y_{(j+1)}$, we obtain

$$\begin{aligned} f_{L_j,Z}(l_j, z) &= \left|\frac{\partial(y_{(j)}, y_{(j+1)})}{\partial(l_j, z)}\right| \cdot f_{Y_{(j)},Y_{(j+1)}}(y_{(j)}, y_{(j+1)}) \\ &= 1 \cdot f_{Y_{(j)},Y_{(j+1)}}(z, z + l_{(j)}) \\ &= \frac{m!}{(j-1)!(m-j-1)!} z^{j-1}(1-l_j-z)^{m-j-1}, \end{aligned}$$

thus

$$f_{L_j}(l_j) = \frac{m!}{(j-1)!(m-j-1)!}\int_0^{1-l_j} z^{j-1}(1-l_j-z)^{m-j-1}dz.$$

By letting $\tilde{z} = z/(1-l_j)$, the above becomes

$$\begin{aligned} f_{L_j}(l_j) &= \frac{m!}{(j-1)!(m-j-1)!}\int_0^1 \big((1-l_j)\cdot\tilde{z}\big)^{j-1}\big((1-l_j)\cdot(1-\tilde{z})\big)^{m-j-1}d\tilde{z} \\ &= \frac{m!(1-l_j)^{m-1}}{(j-1)!(m-j-1)!}\int_0^1 \tilde{z}^{j-1}(1-\tilde{z})^{m-j-1}d\tilde{z}. \end{aligned}$$

Recalling the standard Beta function $B(p,q)$ is defined as $\int_0^1 u^{p-1}(1-u)^{q-1}du = \frac{(p-1)!(q-1)!}{(p+q-1)!}$, we obtain

$$\begin{aligned} f_{L_j}(l_j) &= \frac{m!(1-l_j)^{m-1}}{(j-1)!(m-j-1)!}\cdot\frac{(j-1)!(m-j-1)!}{(m-1)!} \\ &= m(1-l_j)^{m-1}. \end{aligned}$$

Similarly, for any $j_1 < j_2$, we have

$$\begin{aligned}
& f_{Y_{(j_1)},Y_{(j_1+1)},Y_{(j_2)},Y_{(j_2+1)}}(y_{(j_1)}, y_{(j_1+1)}, y_{(j_2)}, y_{(j_2+1)}) \\
=& \int \cdots \int f(y_{(1)}, \ldots, y_{(n)})\, dy_{(1)} \ldots dy_{(j_1-1)}\, dy_{(j_1+2)} \ldots dy_{(j_2-1)}\, dy_{(j_2+2)} \ldots dy_{(m)} \\
=& \underset{0 \le y_{(1)} < \ldots < y_{(m)} \le 1}{\int \cdots \int} m!\, dy_{(1)} \ldots dy_{(j_1-1)}\, dy_{(j_1+2)} \ldots dy_{(j_2-1)}\, dy_{(j_2+2)} \ldots dy_{(m)} \\
=& \frac{m!}{(j_1-1)!(j_2-j_1-2)!(m-j_2-1)!} y_{(j_1)}^{j_1-1} (y_{(j_2)} - y_{(j_1+1)})^{j_2-j_1-2} (1 - y_{(j_2+1)})^{m-j_2-1}.
\end{aligned}$$

If we let $l_{j_1} = y_{(j_1+1)} - y_{(j_1)}, l_{j_2} = y_{(j_2+1)} - y_{(j_2)}$, by again transforming the distribution, we get

$$\begin{aligned}
& f_{L_{j_1},L_{j_2},Y_{(j_1)},Y_{(j_2+1)}}(l_{j_1}, l_{j_2}, y_{(j_1)}, y_{(j_2+1)}) \\
=& \left| \frac{\partial(y_{(j_1)}, y_{(j_1+1)}, y_{(j_1)}, y_{(j_2+1)})}{\partial(l_{j_1}, l_{j_2}, y_{(j_1)}, y_{(j_2+1)})} \right| \cdot f_{Y_{(j_1)},Y_{(j_1+1)},Y_{(j_2)},Y_{(j_2+1)}}(y_{(j_1)}, y_{(j_1+1)}, y_{(j_2)}, y_{(j_2+1)}) \\
=& 1 \cdot f_{Y_{(j_1)},Y_{(j_1+1)},Y_{(j_2)},Y_{(j_2+1)}}(y_{(j_1)}, y_{(j_1)} + l_{j_1}, y_{(j_2+1)} - l_{j_2}, y_{(j_2+1)}) \\
=& \frac{m!}{(j_1-1)!(j_2-j_1-2)!(m-j_2-1)!} \cdot y_{(j_1)}^{j_1-1} (y_{(j_2+1)} - y_{(j_1)} - l_{j_1} - l_{j_2})^{j_2-j_1-2} (1 - y_{(j_2+1)})^{m-j_2-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& f_{L_{j_1},L_{j_2}}(l_{j_1}, l_{j_2}) \\
=& \iint f_{L_{j_1},L_{j_2},Y_{(j_1)},Y_{(j_2+1)}}(l_{j_1}, l_{j_2}, y_{(j_1)}, y_{(j_2+1)})\, dy_{(j_1)}\, dy_{(j_2+1)} \\
=& \frac{m!}{(j_1-1)!(j_2-j_1-2)!(m-j_2-1)!} \iint y_{(j_1)}^{j_1-1} (1 - y_{(j_2+1)})^{m-j_2-1} (y_{(j_2+1)} - y_{(j_1)} - l_{j_1} - l_{j_2})^{j_2-j_1-2}\, dy_{(j_1)}\, dy_{(j_2+1)},
\end{aligned}$$

where the integral above is taken over the region $\{y_{(j_1)}, y_{(j_2+1)} : 0 \le y_{(j_1)} \le y_{(j_2+1)} \le 1, y_{(j_2+1)} - y_{(j_1)} \ge l_{j_1} + l_{j_2}\}$. Calculating the integral generates

$$f_{L_{j_1},L_{j_2}}(l_{j_1}, l_{j_2}) = m(m-1)(1 - l_{j_1} - l_{j_2})^{m-2}. \quad \square$$

The following two identities are frequently used in our later calculations.

Corollary 3.1.1. *If* $F_X = F_Y = U[0,1]$, *then*

$$\begin{aligned} P(N_{j,m}=0) &= \frac{m}{m+n}, \\ P(N_{j_1,m}=0, N_{j_2,m}=0) &= \frac{m(m-1)}{(m+n)(m+n-1)}. \end{aligned}$$

Proof. From Proposition 3.1.1, we know that

$$P(N_{j,m}=0 \mid L_{j,m}=l_{j,m}) = (1-l_{j,m})^n.$$

Hence,

$$\begin{aligned} P(N_{j,m}=0) &= \int_0^1 P(N_{j,m}=0 \mid L_{j,m}=l_{j,m}) f_{L_{j,m}}(l_{j,m})\, dl_{j,m} \\ &= \int_0^1 (1-l_{j,m})^n m(1-l_{j,m})^{m-1}\, dl_{j,m}. \end{aligned}$$

Calculating the integration above gives

$$P(N_{j,m}=0) = \frac{m}{m+n}.$$

Similarly, for $j_1 < j_2$ we have

$$\begin{aligned} & P(N_{j_1,m}=0, N_{j_2,m}=0) \\ &= \iint P(N_{j_1,m}{=}0, N_{j_2,m}{=}0 \mid L_{j_1,m}{=}l_{j_1,m}, L_{j_2,m}{=}l_{j_2,m}) f_{L_{j_1,m},L_{j_2,m}}(l_{j_1,m}, l_{j_2,m})\, dl_{j_1,m}\, dl_{j_2,m} \\ &= \iint (1-l_{j_1,m}-l_{j_2,m})^n \cdot m(m-1)(1-l_{j_1,m}-l_{j_2,m})^{m-2}\, dl_{j_1,m}\, dl_{j_2,m} \\ &= \iint m(m-1)(1-l_{j_1,m}-l_{j_2,m})^{m+n-2}\, dl_{j_1,m}\, dl_{j_2,m}, \end{aligned}$$

where the integrating area is $\{l_{j_1}, l_{j_2} : 0 \leq l_{j_1,m} + l_{j_2,m} \leq 1, 0 \leq l_{j_1,m}, l_{j_2,m} \leq 1\}$. Calculating the integral, we get

$$\begin{aligned} & P(N_{j_1,m} = 0, N_{j_2,m} = 0) \\ = & \int_0^1 \int_0^{1-l_{j_1,m}} m(m-1)(1 - l_{j_1,m} - l_{j_2,m})^{m+n-2} dl_{j_2,m} dl_{j_1,m} \\ = & \int_0^1 \frac{m(m-1)}{m+n-1}(1 - l_{j_1,m})^{m+n-1} dl_{j_1,m} \\ = & \frac{m(m-1)}{(m+n)(m+n-1)}. \quad \square \end{aligned}$$

3.2 Variance Expressed as the Expectations of Functions of $N_{j,m}$

3.2.1 Conditional Moments

The next lemma gives the exact formula of the conditional first and second moments of $\alpha_{j,m}$ given $N_{j,m}$.

Lemma 3.2.1. *If $F_X = F_Y = U[0,1]$, then*

$$E(\alpha_{j,m} \mid N_{j,m}) = \begin{cases} 0 & N_{j,m} = 0, j = 0, \cdots, m, \\ 1 & N_{j,m} > 0, j = 0, m, \\ \frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j,m}}} & N_{j,m} > 0, j = 1, \cdots, m-1, \end{cases}$$

$$Var(\alpha_{j,m} \mid N_{j,m}) = \begin{cases} 0 & N_{j,m} = 0, j = 0, \cdots, m, \\ 0 & N_{j,m} > 0, j = 0, m, \\ \frac{20}{81} - \frac{16}{81}\frac{1}{4^{N_{j,m}}} - \frac{256}{81}\frac{1}{4^{2N_{j,m}}} & N_{j,m} > 0, j = 1, \cdots, m-1. \end{cases}$$

Proof. We compute the conditional moments of $\alpha_{j,m}$ according to the conditional distribution formulas given in Theorem 2.1.1.

A) When $N_{j,m} = 0$, since $\alpha_{j,m} \equiv 0$, we have

$$E(\alpha_{j,m} \mid N_{j,m} = 0) = 0 \qquad \text{for } j = 0, \cdots, m,$$

$$Var(\alpha_{j,m} \mid N_{j,m} = 0) = 0 \qquad \text{for } j = 0, \cdots, m.$$

B) When $N_{j,m} > 0$, depending on whether $\alpha_{j,m}$ is an external or internal component, we have the following two sub-cases.

B.1) For external components, since $\alpha_{j,m} \equiv 1$ given $N_{j,m} > 0$, we have

$$E(\alpha_{j,m} \mid N_{j,m} = n_{j,m}) = 1 \qquad \text{for } n_{j,m} > 0, j = 0 \text{ or } m,$$

$$Var(\alpha_{j,m} \mid N_{j,m} = n_{j,m}) = 0 \qquad \text{for } n_{j,m} > 0, j = 0 \text{ or } m.$$

B.2) For internal components, since $\alpha_{j,m} \in \{1, 2\}$ given $N_{j,m} > 0$, we have

$$\begin{aligned} E(\alpha_{j,m} \mid N_{j,m} = n_{j,m}) &= 1 \cdot P(\alpha_{j,m} = 1 \mid N_{j,m} = n_{j,m}) \\ &\quad + 2 \cdot P(\alpha_{j,m} = 2 \mid N_{j,m} = n_{j,m}). \end{aligned}$$

Plugging the formulas in Theorem 2.1.1 into the equation above yields

$$\begin{aligned} E(\alpha_{j,m} \mid N_{j,m} = n_{j,m}) &= 1 \cdot \left(\frac{5}{9} + \frac{4}{9} \frac{1}{4^{n_{j,m}-1}} \right) \\ &\quad + 2 \cdot \left(\frac{4}{9} - \frac{4}{9} \frac{1}{4^{n_{j,m}-1}} \right) \\ &= \frac{13}{9} - \frac{4}{9} \frac{1}{4^{n_{j,m}-1}}; \end{aligned}$$

thus,

$$E(\alpha_{j,m} \mid N_{j,m} = n_{j,m}) = \frac{13}{9} - \frac{16}{9} \frac{1}{4^{n_{j,m}}} \quad \text{for } n_{j,m} > 0, j = 1, \cdots, m-1.$$

Similarly, the second conditional moment can be calculated as

$$\begin{aligned} E(\alpha_{j,m}^2 \mid N_{j,m} = n_{j,m}) &= 1^2 \cdot P(\alpha_{j,m} = 1 \mid N_{j,m} = n_{j,m}) \\ &\quad + 2^2 \cdot P(\alpha_{j,m} = 2 \mid N_{j,m} = n_{j,m}). \end{aligned}$$

Applying Theorem 2.1.1, the above becomes

$$\begin{aligned} E(\alpha_{j,m}^2 \mid N_{j,m} = n_{j,m}) &= 1 \cdot \left(\frac{5}{9} + \frac{4}{9} \frac{1}{4^{n_{j,m}-1}} \right) \\ &\quad + 4 \cdot \left(\frac{4}{9} - \frac{4}{9} \frac{1}{4^{n_{j,m}-1}} \right) \\ &= \frac{7}{3} - \frac{4}{3} \frac{1}{4^{n_{j,m}-1}}; \end{aligned}$$

thus,

$$E(\alpha_{j,m}^2 \mid N_{j,m} = n_{j,m}) = \frac{7}{3} - \frac{16}{3} \frac{1}{4^{n_{j,m}}} \quad \text{for } n_{j,m} > 0, j = 1, \cdots, m-1.$$

It immediately follows that

$$\begin{aligned} Var(\alpha_{j,m} \mid N_{j,m}) &= E(\alpha_{j,m}^2 \mid N_{j,m}) - \big(E(\alpha_{j,m} \mid N_{j,m}) \big)^2 \\ &= \frac{7}{3} - \frac{16}{3} \frac{1}{4^{N_{j,m}}} - \left(\frac{13}{9} - \frac{16}{9} \frac{1}{4^{N_{j,m}}} \right)^2. \end{aligned}$$

By expanding the quadratic term and combining similar terms, we get

$$
\begin{aligned}
&Var(\alpha_{j,m} \mid N_{j,m}) \\
&= \frac{7}{3} - \frac{16}{3}\frac{1}{4^{N_{j,m}}} - \left(\frac{169}{81} - \frac{416}{81}\frac{1}{4^{N_{j,m}}} + \frac{256}{81}\frac{1}{4^{2N_{j,m}}}\right) \\
&= \frac{20}{81} - \frac{16}{81}\frac{1}{4^{N_{j,m}}} - \frac{256}{81}\frac{1}{4^{2N_{j,m}}}. \quad \square
\end{aligned}
$$

3.2.2 Variance of the Individual Component

Using some of the distribution facts obtained in Section 3.1, we can further calculate $Var(\alpha_{j,m})$. The result is given in the following lemma.

Lemma 3.2.2. *If $F_X = F_Y = U[0,1]$, then*

$$
Var(\alpha_{j,m}) = \begin{cases} \frac{mn}{(m+n)^2} & j = 0, m, \\ \frac{20}{81}\frac{n}{m+n} + \frac{169}{81}\frac{mn}{(m+n)^2} - \left(\frac{16}{81} + \frac{416}{81}\frac{m}{m+n}\right)\mu_{n,m} - \frac{256}{81}\mu_{n,m}{}^2 & j = 1, \cdots, m-1, \end{cases}
$$

where $\mu_{n,m} \equiv E\left(\frac{1}{4^{N_{j,m}}} I_{\{N_{j,m}>0\}}\right)$.

Proof. Since $Var(\alpha_{j,m}) = E\big[Var(\alpha_{j,m} \mid N_{j,m})\big] + Var\big[E(\alpha_{j,m} \mid N_{j,m})\big]$, we write $Var(\alpha_{j,m})$ as

$$
Var(\alpha_{j,m}) = E\big[Var(\alpha_{j,m} \mid N_{j,m})\big] + E\big[E(\alpha_{j,m} \mid N_{j,m})^2\big] - \Big(E\big[E(\alpha_{j,m} \mid N_{j,m})\big]\Big)^2.
$$

Given that $1 = I_{\{N_{j,m}>0\}} + I_{\{N_{j,m}=0\}}$, the equation above can be rewritten as

$$
\begin{aligned}
Var(\alpha_{j,m}) &= E\big[Var(\alpha_{j,m} \mid N_{j,m})(I_{\{N_{j,m}>0\}} + I_{\{N_{j,m}=0\}})\big] \\
&\quad + E\big[E(\alpha_{j,m} \mid N_{j,m})^2(I_{\{N_{j,m}>0\}} + I_{\{N_{j,m}=0\}})\big] \\
&\quad - \Big(E\big[E(\alpha_{j,m} \mid N_{j,m})(I_{\{N_{j,m}>0\}} + I_{\{N_{j,m}=0\}})\big]\Big)^2,
\end{aligned}
$$

thus by linearity,

$$\begin{aligned}
Var(\alpha_{j,m}) &= E\big[Var(\alpha_{j,m} \mid N_{j,m}) I_{\{N_{j,m}>0\}}\big] + E\big[Var(\alpha_{j,m} \mid N_{j,m}) I_{\{N_{j,m}=0\}}\big] \\
&\quad + E\big[E(\alpha_{j,m} \mid N_{j,m})^2 I_{\{N_{j,m}>0\}}\big] + E\big[E(\alpha_{j,m} \mid N_{j,m})^2 I_{\{N_{j,m}=0\}}\big] \\
&\quad - \Big(E\big[E(\alpha_{j,m} \mid N_{j,m}) I_{\{N_{j,m}>0\}}\big] + E\big[E(\alpha_{j,m} \mid N_{j,m}) I_{\{N_{j,m}=0\}}\big]\Big)^2.
\end{aligned}$$

Recall that $E(\alpha_{j,m} \mid N_{j,m} = 0) = 0$ and $Var(\alpha_{j,m} \mid N_{j,m} = 0) = 0$ (Lemma 3.2.1), the equation above can be simplified as

$$\begin{aligned}
Var(\alpha_{j,m}) &= E\big[Var(\alpha_{j,m} \mid N_{j,m}) I_{\{N_{j,m}>0\}}\big] \\
&\quad + E\big[E(\alpha_{j,m} \mid N_{j,m})^2 I_{\{N_{j,m}>0\}}\big] \\
&\quad - \Big(E\big[E(\alpha_{j,m} \mid N_{j,m}) I_{\{N_{j,m}>0\}}\big]\Big)^2.
\end{aligned}$$

By substituting the formulas given in Lemma 3.2.1, we simplify the equations above as follows:

A) When $\alpha_{j,m}$ is an external component,

$$\begin{aligned}
Var(\alpha_{j,m}) &= E(I_{\{N_{j,m}>0\}}) - \big(E(I_{\{N_{j,m}>0\}})\big)^2 \\
&= P(N_{j,m} > 0) - \big(P(N_{j,m} > 0)\big)^2 \\
&= P(N_{j,m} = 0)\big(1 - P(N_{j,m} = 0)\big);
\end{aligned}$$

thus, applying Corollary 3.1.1 gives

$$Var(\alpha_{j,m}) = \frac{mn}{(m+n)^2}.$$

B) When $\alpha_{j,m}$ is an internal component,

$$\begin{aligned} Var(\alpha_{j,m}) &= E\left[\left(\frac{20}{81} - \frac{16}{81}\frac{1}{4^{N_{j,m}}} - \frac{256}{81}\frac{1}{4^{2N_{j,m}}}\right) I_{\{N_{j,m}>0\}}\right] \\ &+ E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j,m}}}\right)^2 I_{\{N_{j,m}>0\}}\right] \\ &- \left(E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j,m}}}\right) I_{\{N_{j,m}>0\}}\right]\right)^2. \end{aligned}$$

Expanding the terms on the RHS of the equation above yields

$$\begin{aligned} & Var(\alpha_{j,m}) \\ &= \frac{20}{81}P(N_{j,m}>0) - \frac{16}{81}E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) - \frac{256}{81}E\left(\frac{1}{4^{2N_{j,m}}}I_{\{N_{j,m}>0\}}\right) \\ &+ \frac{169}{81}P(N_{j,m}>0) - \frac{416}{81}E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) + \frac{256}{81}E\left(\frac{1}{4^{2N_{j,m}}}I_{\{N_{j,m}>0\}}\right) \\ &- \frac{169}{81}\left(P(N_{j,m}>0)\right)^2 + \frac{416}{81}P(N_{j,m}>0)E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) - \frac{256}{81}\left[E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right)\right]^2, \end{aligned}$$

and collecting similar terms generates

$$\begin{aligned} Var(\alpha_{j,m}) &= \frac{20}{81} - \frac{20}{81}P(N_{j,m}=0) - \frac{16}{81}E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) \\ &+ \frac{169}{81}P(N_{j,m}=0)\big(1 - P(N_{j,m}=0)\big) \\ &- \frac{416}{81}P(N_{j,m}=0)E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) \\ &- \frac{256}{81}\left[E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right)\right]^2. \end{aligned}$$

Again, by applying Corollary 3.1.1, the equation above reduces to

$$\begin{aligned} Var(\alpha_{j,m}) &= \frac{20}{81} - \frac{20}{81}\frac{m}{m+n} - \frac{16}{81}E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) \\ &+ \frac{169}{81}\frac{mn}{(m+n)^2} \\ &- \frac{416}{81}\frac{m}{m+n}E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right) \\ &- \frac{256}{81}\left[E\left(\frac{1}{4^{N_{j,m}}}I_{\{N_{j,m}>0\}}\right)\right]^2. \end{aligned}$$

Collecting similar terms on the RHS of the above finishes the proof. □

3.2.3 Covariance Between Components

Lemma 3.2.3. *If $F_X = F_Y = U[0,1]$, then the covariance between any two components is given by*

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = \begin{cases} -\frac{mn}{(m+n)^2(m+n-1)} & j_1 = 0, j_2 = m \\ -\frac{13}{9}\frac{mn}{(m+n)^2(m+n-1)} - \frac{16}{9}\delta_{n,m} & j_1 = 0, j_2 = 1, \cdots, m-1 \\ -\frac{169}{81}\frac{mn}{(m+n)^2(m+n-1)} - \frac{416}{81}\delta_{n,m} + \frac{256}{81}(\nu_{n,m} - {\mu_{n,m}}^2) & j_1, j_2 = 1, \cdots, m-1, \end{cases}$$

where

$$\begin{aligned} \delta_{n,m} &\equiv E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - \frac{n}{m+n} E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right), \\ \nu_{n,m} &\equiv E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right). \end{aligned}$$

Proof. We first express the covariance in terms of the conditional expectation as follows:

$$\begin{aligned} Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) &= E(\alpha_{j_1,m}\alpha_{j_2,m}) - E(\alpha_{j_1,m})E(\alpha_{j_2,m}) \\ &= E\big[E(\alpha_{j_1,m}\alpha_{j_2,m} \mid N_{j_1,m}, N_{j_2,m})\big] \\ &\quad - E\big[E(\alpha_{j_1,m} \mid N_{j_1,m})\big]E\big[E(\alpha_{j_2,m} \mid N_{j_2,m})\big]. \end{aligned}$$

From Proposition 3.1.2, the last equality can be further written as

$$\begin{aligned} Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) &= E\big[E(\alpha_{j_1,m} \mid N_{j_1,m}, N_{j_2,m})E(\alpha_{j_2,m} \mid N_{j_1,m}, N_{j_2,m})\big] \\ &\quad - E\big[E(\alpha_{j_1,m} \mid N_{j_1,m})\big]E\big[E(\alpha_{j_2,m} \mid N_{j_2,m})\big] \\ &= E\big[E(\alpha_{j_1,m} \mid N_{j_1,m})E(\alpha_{j_2,m} \mid N_{j_2,m})\big] \\ &\quad - E\big[E(\alpha_{j_1,m} \mid N_{j_1,m})]E[E(\alpha_{j_2,m} \mid N_{j_2,m})\big]. \end{aligned}$$

By substituting the formulas obtained in Lemma 3.2.1, we simplify the equation above in different cases as follows:

A) When $\alpha_{j_1,m}$ and $\alpha_{j_2,m}$ are both external components,

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = E\big(I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\big) - \Big[E\big(I_{\{N_{j_2,m}>0\}}\big)\Big]^2.$$

Since $N_{j_1,m}$ and $N_{j_2,m}$ are non-negative,

$$1 = I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}} + I_{\{N_{j_1,m}=0\}} + I_{\{N_{j_2,m}=0\}} - I_{\{N_{j_1,m}=0, N_{j_2,m}=0\}}.$$

Hence,

$$\begin{aligned} &Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) \\ &= 1 - E(I_{\{N_{j_1,m}=0\}}) - E(I_{\{N_{j_2,m}=0\}}) + E(I_{\{N_{j_1,m}=0, N_{j_2,m}=0\}}) - \big[E(I_{\{N_{j_2,m}>0\}})\big]^2 \\ &= 1 - 2P(N_{j_2,m}=0) + P(N_{j_1,m}=0, N_{j_2,m}=0) - \big(1 - P(N_{j_2,m}=0)\big)^2. \end{aligned}$$

Applying Corollary 3.1.1 reduces the equation above to

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = 1 - 2\frac{m}{m+n} + \frac{m(m-1)}{(m+n)(m+n-1)} - \left(1 - \frac{m}{m+n}\right)^2.$$

If we expand the quadratic term and then cancel similar terms, it follows that

$$
\begin{aligned}
& Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) \\
= \; & 1 - 2\frac{m}{m+n} + \frac{m(m-1)}{(m+n)(m+n-1)} - \left(1 - 2\frac{m}{m+n} + \left(\frac{m}{m+n}\right)^2\right) \\
= \; & \frac{m(m-1)}{(m+n)(m+n-1)} - (\frac{m}{m+n})^2.
\end{aligned}
$$

Reducing the factions above to a common denominator gives

$$
Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = -\frac{mn}{(m+n)^2(m+n-1)}.
$$

B) When $\alpha_{j_1,m}$ is an external component while $\alpha_{j_2,m}$ is an internal component,

$$
\begin{aligned}
Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = \; & E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j_2,m}}}\right) I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right] \\
& - E\left(I_{\{N_{j_1,m}>0\}}\right) E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j_2,m}}}\right) I_{\{N_{j_2,m}>0\}}\right].
\end{aligned}
$$

Through expansion of the terms in the expectation, the above is simplified as

$$
\begin{aligned}
& Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) \\
= & \frac{13}{9} E\left(I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - \frac{16}{9} E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) \\
& - \frac{13}{9}\left[E\left(I_{\{N_{j_2,m}>0\}}\right)\right]^2 + \frac{16}{9} E\left(I_{\{N_{j_1,m}>0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right) \\
= & \frac{13}{9}\left(E\left(I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - \left[E\left(I_{\{N_{j_2,m}>0\}}\right)\right]^2\right) \\
& - \frac{16}{9}\left[E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - E\left(I_{\{N_{j_1,m}>0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right)\right].
\end{aligned}
$$

Recall that in the previous case we obtained

$$
E\left(I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - \left[E\left(I_{\{N_{j_2,m}>0\}}\right)\right]^2 = -\frac{mn}{(m+n)^2(m+n-1)},
$$

and that by Corollary 3.1.1 we have

$$E\left(I_{\{N_{j_2,m}>0\}}\right) = P\left(N_{j_2,m}>0\right) = \frac{n}{m+n}.$$

Therefore, we further write the covariance as

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = -\frac{13}{9}\frac{mn}{(m+n)^2(m+n-1)} - \frac{16}{9}\delta_{n,m},$$

where

$$\delta_{n,m} \equiv E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - \frac{n}{m+n} E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right).$$

C) When $\alpha_{j_1,m}$ and $\alpha_{j_2,m}$ are both internal components,

$$\begin{aligned}
& Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) \\
= \; & E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j_1,m}}}\right)\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j_2,m}}}\right) I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right] \\
& - E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j_1,m}}}\right) I_{\{N_{j_1,m}>0\}}\right] E\left[\left(\frac{13}{9} - \frac{16}{9}\frac{1}{4^{N_{j_2,m}}}\right) I_{\{N_{j_2,m}>0\}}\right].
\end{aligned}$$

Expanding the expressions inside the expectations above gives

$$\begin{aligned}
& Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) \\
= & \left(\frac{13}{9}\right)^2 E\left(I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - 2\cdot\frac{13}{9}\cdot\frac{16}{9} E\left(\frac{1}{4^{N_{j_1,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) \\
& \qquad + \left(\frac{16}{9}\right)^2 E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) \\
& - \left(\frac{13}{9}\right)^2 \left[E\left(I_{\{N_{j_2,m}>0\}}\right)\right]^2 + 2\cdot\frac{13}{9}\cdot\frac{16}{9} E\left(I_{\{N_{j_2,m}>0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right) \\
& \qquad - \left(\frac{16}{9}\right)^2 \left[E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right)\right]^2.
\end{aligned}$$

Collecting terms simplifies the above to

$$
\begin{aligned}
&Cov(\alpha_{j_1,m},\alpha_{j_2,m})\\
=\;&\left(\frac{13}{9}\right)^2\left(E\left(I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}}\right)-\left[E(I_{\{N_{j_2,m}>0\}})\right]^2\right)\\
&-2\cdot\frac{13}{9}\cdot\frac{16}{9}\left[E\left(\frac{1}{4^{N_{j_2,m}}}I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}}\right)-E(I_{\{N_{j_2,m}>0\}})E\left(\frac{1}{4^{N_{j_2,m}}}I_{\{N_{j_2,m}>0\}}\right)\right]\\
&+\left(\frac{16}{9}\right)^2\left(E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}}I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}}\right)-\left[E\left(\frac{1}{4^{N_{j_2,m}}}I_{\{N_{j_2,m}>0\}}\right)\right]^2\right).
\end{aligned}
$$

Denote

$$\nu_{n,m} \quad = \quad E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}}I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}}\right).$$

By recalling

$$E\left(I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}}\right)-\left[E\left(I_{\{N_{j_2,m}>0\}}\right)\right]^2=-\frac{mn}{(m+n)^2(m+n-1)}$$

as obtained in the previous case, we have

$$
\begin{aligned}
&Cov(\alpha_{j_1,m},\alpha_{j_2,m})\\
=\;&-\frac{169}{81}\frac{mn}{(m+n)^2(m+n-1)}-\frac{416}{81}\delta_{n,m}+\frac{256}{81}\left(\nu_{n,m}-{\mu_{n,m}}^2\right)
\end{aligned}
$$

as desired. □

3.2.4 Summary

In the very beginning of this chapter, we decomposed the variance of the domination number as follows:

$$
\begin{aligned}
Var(\Gamma_{n,m}) &= 2Var(\alpha_{0,m})+(m-1)Var(\alpha_{1,m})\\
&+2\;Cov(\alpha_{0,m},\alpha_{m,m})+2(m-1)Cov(\alpha_{0,m},\alpha_{1,m})+m(m-1)Cov(\alpha_{1,m},\alpha_{2,m}).
\end{aligned}
$$

In Lemmas 3.2.2 and 3.2.3 in this section, we proved that

$$Var(\alpha_{j,m}) = \begin{cases} \frac{mn}{(m+n)^2} & j = 0, m \\ \frac{20}{81}\frac{n}{m+n} + \frac{169}{81}\frac{mn}{(m+n)^2} - \left(\frac{16}{81} + \frac{416}{81}\frac{m}{m+n}\right)\mu_{n,m} - \frac{256}{81}{\mu_{n,m}}^2 & j = 1, \cdots, m-1, \end{cases}$$

and

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = \begin{cases} -\frac{mn}{(m+n)^2(m+n-1)} & j_1 = 0, j_2 = m \\ -\frac{13}{9}\frac{mn}{(m+n)^2(m+n-1)} - \frac{16}{9}\delta_{n,m} & j_1 = 0, j_2 = 1, \cdots, m-1 \\ -\frac{169}{81}\frac{mn}{(m+n)^2(m+n-1)} - \frac{416}{81}\delta_{n,m} + \frac{256}{81}\left(\nu_{n,m} - {\mu_{n,m}}^2\right) & j_1, j_2 = 1, \cdots, m-1, \end{cases}$$

where

$$\begin{aligned} \mu_{n,m} &\equiv E\left(\frac{1}{4^{N_{j,m}}} I_{\{N_{j,m}>0\}}\right), \\ \delta_{n,m} &\equiv E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - \frac{n}{m+n} E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right), \\ \nu_{n,m} &\equiv E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right). \end{aligned}$$

Therefore, $Var(\Gamma_{n,m})$ can be expressed in terms of $\mu_{n,m}, \delta_{n,m}$ and $\nu_{n,m}$, which are expectations of some exponential functions involving $N_{j,m}$.

3.3 Variance Expressed in the Summation Form in Terms of n and m

We have proved that $N_{j,m}, j = 1, \cdots, m$, are multinomially distributed conditioned on $L_{j,m}, j = 1, \cdots, m$, and in Proposition 3.1.3 we obtained the distribution of $L_{j,m}$. Therefore, theoretically, we know the distribution of $N_{j,m}$, using which we can integrate the expectations into the formulas for $\mu_{n,m}, \delta_{n,m}$ and $\nu_{n,m}$. In this section, we calculate the explicit summation forms of $\mu_{n,m}, \delta_{n,m}$ and $\nu_{n,m}$.

3.3.1 Component Variance Expressed in the Summation Form in Terms of n and m

From Lemma 3.2.2, we know that each component's variance is determined by $\mu_{n,m}$. In this sub-section, we convert $\mu_{n,m}$ into a summation form.

Given that $\mu_{n,m}$ is defined as $E\left(\frac{1}{4^{N_{j,m}}} I_{\{N_{j,m}>0\}}\right)$, it follows that

$$\begin{aligned} \mu_{n,m} &= E\left[\frac{1}{4^{N_{j,m}}}\left(I_{\{N_{j,m}\geq 0\}} - I_{\{N_{j,m}=0\}}\right)\right] \\ &= E\left(\frac{1}{4^{N_{j,m}}}\right) - P\left(N_{j,m}=0\right). \end{aligned}$$

From Proposition 3.1.1, we have

$$P\left(N_{j,m}=q \mid L_{j,m}=l_{j,m}\right) = \binom{n}{q} l_{j,m}^{q}(1-l_{j,m})^{n-q},$$

so

$$E\left(\frac{1}{4^{N_{j,m}}} \mid L_{j,m} = l_{j,m}\right) = \sum_{q=0}^{n} \frac{1}{4^q}\binom{n}{q} l_{j,m}^q (1 - l_{j,m})^{n-q}.$$

Applying Corollary 3.1.1 yields

$$\begin{aligned}\mu_{n,m} &= E\left[E\left(\frac{1}{4^{N_{j,m}}} \mid L_{j,m}\right)\right] - P\left(N_{j,m} = 0\right) \\ &= E\left[\sum_{q=0}^{n} \frac{1}{4^q}\binom{n}{q} L_{j,m}^q (1 - L_{j,m})^{n-q}\right] - \frac{m}{m+n}.\end{aligned}$$

Using the distribution of $L_{j,m}$ as given in Proposition 3.1.3, the last expectation can be written in the integration form as

$$\mu_{n,m} = \int_0^1 \sum_{q=0}^{n} \frac{1}{4^q}\binom{n}{q} {l_{j,m}}^q (1 - l_{j,m})^{n-q} m (1 - l_{j,m})^{m-1}\, dl_{j,m} - \frac{m}{m+n}.$$

When the integration and summation operations are exchanged, the above becomes

$$\mu_{n,m} = \sum_{q=0}^{n} \int_0^1 \frac{1}{4^q}\binom{n}{q} {l_{j,m}}^q (1 - l_{j,m})^{n-q} m (1 - l_{j,m})^{m-1}\, dl_{j,m} - \frac{m}{m+n}.$$

Computing the integrals above gives

$$\mu_{n,m} = m \sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \binom{n}{q} B(q+1, m+n-q) - \frac{m}{m+n},$$

where $B(i,j)$ is the standard beta function with value $\frac{(i-1)!(j-1)!}{(i+j-1)!}$ when i and j are positive integers. Therefore,

$$\mu_{n,m} = m \sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \frac{n!}{q!(n-q)!} \frac{q!(m+n-q-1)!}{(m+n)!} - \frac{m}{m+n}.$$

By cancelling the factor $q!$ inside the summation and factoring out $\frac{m}{m+n}$ from the entire expression, we finally obtain

$$\mu_{n,m} = \frac{m}{m+n}\left(\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!} - 1\right). \qquad (3.3.1)$$

3.3.2 Component Covariance Expressed in the Summation Form in Terms of n and m

From Lemma 3.2.2, we know that the covariance between any two components is determined by $\mu_{n,m}, \delta_{n,m}$ and $\nu_{n,m}$. In this sub-section, we convert $\delta_{n,m}$ and $\nu_{n,m}$ into a summation form.

3.3.2.1 $\delta_{n,m}$ Expressed in the Summation Form in Terms of n and m

Recall that $\delta_{n,m}$ is defined as

$$\delta_{n,m} = E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}>0, N_{j_2,m}>0\}}\right) - E\left(I_{\{N_{j_2,m}>0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}>0\}}\right).$$

Since $I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}} + I_{\{N_{j_1,m}=0\}} + I_{\{N_{j_2,m}=0\}} - I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}} \equiv 1$, we have

$$\begin{aligned}\delta_{n,m} &= E\left[\frac{1}{4^{N_{j_2,m}}}\left(1 - I_{\{N_{j_1,m}=0\}} - I_{\{N_{j_2,m}=0\}} + I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}}\right)\right] \\ &\quad -E\left(1 - I_{\{N_{j_2,m}=0\}}\right) E\left[\frac{1}{4^{N_{j_2,m}}}\left(1 - I_{\{N_{j_2,m}=0\}}\right)\right].\end{aligned}$$

Expanding terms in the equality above gives

$$\begin{aligned}&\delta_{n,m} \\ &= E\left(\frac{1}{4^{N_{j_2,m}}}\right) - E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0\}}\right) - E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}=0\}}\right) + E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}}\right) \\ &\quad - E\left(\frac{1}{4^{N_{j_2,m}}}\right) + E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}=0\}}\right) + E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}}\right) - E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}=0\}}\right).\end{aligned}$$

Collecting similar terms yields

$$\begin{aligned}\delta_{n,m} = &- E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0\}}\right) + E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}}\right) \\ &+ E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}}\right) - E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}=0\}}\right). \quad (3.3.2)\end{aligned}$$

By conditioning on $L_{j_1,m}$ and $L_{j_2,m}$, the first term at the RHS of (3.3.2) becomes

$$E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0\}}\right) = E\left[E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0\}} \mid L_{j_1,m}, L_{j_2,m}\right)\right].$$

From the distribution of $N_{j,m}$ as given in Proposition 3.1.1, the above becomes

$$E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0\}}\right) = E\left[\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \binom{n}{q} (L_{j_2,m})^q (1 - L_{j_1,m} - L_{j_2,m})^{n-q}\right].$$

Similarly, the third term in (3.3.2) can be written as

$$\begin{aligned} & E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}}\right) \\ = \; & P\left(N_{j_2,m} = 0\right) E\left[\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \binom{n}{q} (L_{j_2,m})^q (1 - L_{j_2,m})^{n-q}\right]. \end{aligned}$$

By applying Corollary 3.1.1, the last equality becomes

$$E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}}\right) = \frac{m}{m+n} E\left[\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \binom{n}{q} (L_{j_2,m})^q (1 - L_{j_2,m})^{n-q}\right].$$

The second and fourth terms in Equation (3.3.2) can be calculated as

$$E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_1,m}=0, N_{j_2,m}=0\}}\right) = P\left(N_{j_1,m} = 0, N_{j_2,m} = 0\right) = \frac{m(m-1)}{(m+n)(m+n-1)},$$

and

$$E\left(I_{\{N_{j_2,m}=0\}}\right) E\left(\frac{1}{4^{N_{j_2,m}}} I_{\{N_{j_2,m}=0\}}\right) = \left[P\left(N_{j_2,m} = 0\right)\right]^2 = \left(\frac{m}{m+n}\right)^2.$$

Therefore, Equation (3.3.2) becomes

$$\begin{aligned} \delta_{n,m} = & - E\left[\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \binom{n}{q} (L_{j_2,m})^q (1 - L_{j_2,m} - L_{j_2,m})^{n-q}\right] \\ & + \frac{m}{m+n} E\left[\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \binom{n}{q} (L_{j_2,m})^q (1 - L_{j_2,m})^{n-q}\right] \\ & + \frac{m(m-1)}{(m+n)(m+n-1)} - \left(\frac{m}{m+n}\right)^2. \end{aligned}$$

The two expectations above can be calculated as

$$\begin{aligned}
& E\left[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}(L_{j_2,m})^{q}\left(1-L_{j_2,m}-L_{j_2,m}\right)^{n-q}\right] \\
= & \sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}\int_0^1\int_0^{1-l_{j_1,m}} l_{j_2,m}^{q}(1-l_{j_1,m}-l_{j_2,m})^{n-q} \\
& \qquad \cdot m(m-1)(1-l_{j_1,m}-l_{j_2,m})^{m-2}dl_{j_2,m}dl_{j_1,m} \\
\overset{z=\frac{l_{j_2,m}}{1-l_{j_1,m}}}{=} & m(m-1)\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}\int_0^1(1-l_{j_1,m})^{m+n-1}\,dl_{j_1,m}\int_0^1 z^{q}(1-z)^{m+n-q-2}dz \\
= & m(m-1)\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}\frac{1}{m+n}B(q+1,m+n-q-1),
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}(L_{j_2,m})^{q}(1-L_{j_2,m})^{n-q}\right] \\
= & \sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}\int_0^1 l_{j_2,m}^{q}(1-l_{j_2,m})^{n-q}m(1-l_{j_2,m})^{m-1}dl_{j_2,m} \\
= & m\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}B(q+1,m+n-q).
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta_{n,m} = & -m(m-1)\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}\frac{1}{m+n}B(q+1,m+n-q-1) \\
& +\frac{m}{m+n}m\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}B(q+1,m+n-q) \\
& +\frac{m(m-1)}{(m+n)(m+n-1)}-\left(\frac{m}{m+n}\right)^{2}.
\end{aligned}$$

By considering $B(x,y) = \frac{(x+y-1)!}{(x-1)!(y-1)!}$, it follows that

$$
\begin{aligned}
\delta_{n,m} &= -m(m-1)\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\frac{n!}{(n-q)!q!}\frac{q!(m+n-q-2)!}{(m+n)!} \\
&\quad + \frac{m^2}{m+n}\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\frac{n!}{(n-q)!q!}\frac{q!(m+n-q-1)!}{(m+n)!} \\
&\quad + \frac{m(m-1)}{(m+n)(m+n-1)} - \left(\frac{m}{m+n}\right)^2.
\end{aligned}
$$

Collecting similar terms gives

$$
\begin{aligned}
\delta_{n,m} &= \sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\frac{n!(m+n-q-2)!}{(n-q)!(m+n)!}\left(\frac{m^2(m+n-q-1)}{m+n} - m(m-1)\right) \\
&\quad - \frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}
$$

Thus,

$$
\delta_{n,m} = \sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\frac{n!(m+n-q-2)!}{(n-q)!(m+n)!}\frac{mn-m^2q}{m+n} - \frac{mn}{(m+n)^2(m+n-1)}. \tag{3.3.3}
$$

3.3.2.2 $\nu_{n,m} - {\mu_{n,m}}^2$ Expressed in the Summation Form in Terms of n and m

Since $I_{\{N_{j_1,m}>0,N_{j_2,m}>0\}} + I_{\{N_{j_1,m}=0\}} + I_{\{N_{j_2,m}=0\}} - I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}} \equiv 1$, we get

$$
\begin{aligned}
\nu_{n,m} &= E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}}\right) - E\left(\frac{1}{4^{N_{j_2,m}}}I_{\{N_{j_1,m}=0\}}\right) - E\left(\frac{1}{4^{N_{j_1,m}}}I_{\{N_{j_2,m}=0\}}\right) \\
&\quad + E\left(I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}}\right).
\end{aligned}
$$

In the previous section, we found that the second term is equal to

$$
E\left(\frac{1}{4^{N_{j_2,m}}}I_{\{N_{j_1,m}=0\}}\right) = m(m-1)\sum_{q=0}^{n}\left(\frac{1}{4}\right)^{q}\binom{n}{q}\frac{1}{m+n}B(q+1,m+n-q-1),
$$

and the third term, by symmetry, is equal to the second term. By Corollary 3.1.1, the fourth term is equal to

$$E\left(I_{\{N_{j_1,m}=0,N_{j_2,m}=0\}}\right) = P\left(N_{j_1,m}=0, N_{j_2,m}=0\right) = \frac{m(m-1)}{(m+n)(m+n-1)}.$$

Next, we focus on calculating the first term in the expression of $\nu_{n,m}$. By applying Proposition 3.1.1, we get

$$\begin{aligned} & E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}\geq 0,N_{j_2,m}\geq 0\}}\right) \\ = \; & E\left[E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}\geq 0,N_{j_2,m}\geq 0\}} \mid L_{j_1,m}, L_{j_2,m}\right)\right] \\ = \; & E\left[\sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \binom{n}{p,q} \left(L_{j_1,m}\right)^p \left(L_{j_2,m}\right)^q \left(1-L_{j_1,m}-L_{j_2,m}\right)^{n-p-q}\right]. \end{aligned}$$

Interchanging the expectation and summation yields

$$\begin{aligned} & E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}\geq 0,N_{j_2,m}\geq 0\}}\right) \\ = \; & \sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \binom{n}{p,q} E\left[\left(L_{j_1,m}\right)^p \left(L_{j_2,m}\right)^q \left(1-L_{j_1,m}-L_{j_2,m}\right)^{n-p-q}\right]. \end{aligned}$$

By plugging in the joint distribution of $L_{j_1,m}$ and $L_{j_2,m}$, the expectation above can be integrated as

$$\begin{aligned} & E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}\geq 0,N_{j_2,m}\geq 0\}}\right) \\ & = m(m-1)\sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \binom{n}{p,q} \int_0^1\int_0^{1-l_{j_1,m}} \left(l_{j_1,m}\right)^p \left(l_{j_2,m}\right)^q \\ & \qquad \cdot \left(1-l_{j_1,m}-l_{j_2,m}\right)^{m+n-p-q-2} \, dl_{j_2,m} dl_{j_1,m}. \end{aligned}$$

A simple substitution of $z = \frac{l_{j_2,m}}{1-l_{j_1,m}}$ yields

$$\begin{aligned}
& E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}\geq 0, N_{j_2,m}\geq 0\}}\right) \\
= \;& m(m-1) \sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \binom{n}{p,q} \int_0^1 {l_{j_1,m}}^p (1-l_{j_1,m})^{m+n-p-1} dl_{j_1,m} \\
& \cdot \int_0^1 z^q (1-z)^{m+n-p-q-2} dz.
\end{aligned}$$

By the definition of a Beta function, the above becomes

$$\begin{aligned}
& E\left(\frac{1}{4^{N_{j_1,m}+N_{j_2,m}}} I_{\{N_{j_1,m}\geq 0, N_{j_2,m}\geq 0\}}\right) \\
= \;& m(m-1) \sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \binom{n}{p,q} B(p+1, m+n-p) B(q+1, m+n-p-q-1).
\end{aligned}$$

Therefore, we can finally write $v_{n,m}$ in the following summation form:

$$\begin{aligned}
& \nu_{n,m} \\
= \;& m(m-1) \sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \binom{n}{p,q} B(p+1, m+n-p) B(q+1, m+n-p-q-1) \\
& - 2\, m(m-1) \sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \binom{n}{p,q} \frac{1}{m+n} B(q+1, m+n-q-1) \\
& + \frac{m(m-1)}{(m+n)(m+n-1)}.
\end{aligned}$$

By factoring out $m(m-1)$ and substituting for the Beta function, we get

$$\begin{aligned}
\nu_{n,m} \;=\; m(m-1)\Bigg\{ & \sum_{\substack{p,q\geq 0\\ p+q\leq n}} \left(\frac{1}{4}\right)^{p+q} \frac{n!(m+n-p-q-2)!}{(n-p-q)!(m+n)!} \\
& - 2 \sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!} \\
& + \frac{1}{(m+n)(m+n-1)} \Bigg\}.
\end{aligned}$$

Let $k = p + q$. Then the first double-sum term in the bracket above can be reduced to a single-sum form as follows:

$$\sum_{k=0}^{n} \left(\frac{1}{4}\right)^{k} \frac{n!(m+n-k-2)!}{(n-k)!(m+n)!}(k+1).$$

Combining the above with the second term in the expression of $\nu_{n,m}$, we get

$$\nu_{n,m} = m(m-1)\left\{\sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!}(q-1) + \frac{1}{(m+n)(m+n-1)}\right\}.$$

This together with Equation (3.3.1) gives

$$\begin{aligned} &\nu_{n,m} - {\mu_{n,m}}^2 \\ = \;& m(m-1)\left\{\sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!}(q-1) + \frac{1}{(m+n)(m+n-1)}\right\} \\ & - \left\{m\sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-1)!}{(n-q)!(m+n)!} - \frac{m}{m+n}\right\}^2. \end{aligned}$$

Multiplying the factor $m(m-1)$ with the two terms inside the first bracket and expanding the quadratic term in the equation above yields

$$\begin{aligned} &\nu_{n,m} - {\mu_{n,m}}^2 \\ &= m(m-1)\sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!}(q-1) + \frac{m(m-1)}{(m+n)(m+n-1)} \\ &\quad -m^2\left(\sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-1)!}{(n-q)!(m+n)!}\right)^2 + \frac{2m^2}{m+n}\sum_{q=0}^{n} \left(\frac{1}{4}\right)^{q} \frac{n!(m+n-q-1)!}{(n-q)!(m+n)!} \\ &\quad - \frac{m^2}{(m+n)^2}. \end{aligned}$$

Combining the first summation with the third summation in the last equality, keeping the second summation, and adding up the two non-summmational terms, we get

$$\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
= \quad & m\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n)!}\left(\frac{(m-1)(q-1)}{m+n-q-1}+\frac{2m}{m+n}\right) \\
& -m^2\left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n)!}\right)^2 \\
& -\frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}$$

A reduction to a common denominator in the final factor in the first summation above produces

$$\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
&= m\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n)!}\frac{(m-1)(q-1)(m+n)+2m(m+n-q-1)}{(m+n)(m+n-q-1)} \\
&- \frac{m^2}{(m+n)^2}\left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!}\right)^2 \\
&- \frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}$$

Cancelling the factor $(m+n-q-1)$ and applying $(m+n)! = (m+n-2)!(m+n)(m+n-1)$ in the first summation gives

$$\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
&= m\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n-2)!}\frac{(m-1)(q-1)(m+n)+2m(m+n-q-1)}{(m+n)^2(m+n-1)} \\
&- \frac{m^2}{(m+n)^2}\left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!}\right)^2 \\
&- \frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}$$

Once we factor out $\frac{m^2}{(m+n)^2}$ from the first two terms in the equation above, it follows that

$$\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
&= \frac{m^2}{(m+n)^2}\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n-2)!}\,\frac{(m-1)(q-1)(m+n)+2m(m+n-q-1)}{m(m+n-1)}\right. \\
&\qquad\left. - \left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!}\right)^2\right\} \\
&- \frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}$$

By using $\frac{(m+n)(m-1)(q-1)+2m(m+n-q-1)}{m(m+n-1)} = q+1+\frac{n-(2m+n)q}{m^2+mn-m}$, the above reduces to

$$\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
&= \frac{m^2}{(m+n)^2}\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n-2)!}\left(q+1+\frac{n-(2m+n)q}{m^2+mn-m}\right)\right. \\
&\qquad\left. - \left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!}\right)^2\right\} \\
&- \frac{mn}{(m+n)^2(m+n-1)}. \qquad (3.3.4)
\end{aligned}$$

3.4 Asymptotic Results

In this section, we show that if $m/n \to r$ as $n \to \infty$, then

$$\begin{aligned}
\mu_{n,m} &= \mu_r + o(1), \\
\delta_{n,m} &= \delta_r \cdot \frac{1}{n} + o\left(\frac{1}{n}\right), \\
\nu_{n,m} - {\mu_{n,m}}^2 &= \nu_r \cdot \frac{1}{n} + o\left(\frac{1}{n}\right),
\end{aligned}$$

where μ_r, δ_r, ν_r are all constants determined by r.

In the calculation of the exact limiting values μ_r, δ_r, ν_r, we rely heavily on the following version of the dominated convergence theorem (DCT).

Theorem 3.4.1. *If $D_n(q) \overset{n\to\infty}{\longrightarrow} D(q)$, and $|D_n(q)| \le D^*(q)$ for all n, where $\sum_{q=0}^{\infty} D^*(q) < \infty$, then*

$$\sum_{q=0}^{\infty} D_n(q) \overset{n\to\infty}{\longrightarrow} \sum_{q=0}^{\infty} D(q).$$

3.4.1 Limiting Value of $\mu_{n,m}$

Recall that in Equation (3.3.1),

$$\mu_{n,m} = \frac{m}{m+n}\left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!} - 1\right),$$

and let $D_n(q) = \begin{cases} \left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!} & q \le n \\ 0 & q > n \end{cases}$. Then $\mu_{n,m}$ can be written as

$$\frac{m}{m+n}\Big(\sum_{q=0}^{\infty} D_n(q) - 1\Big).$$

Recall that $m \equiv m(n)$ and $m/n \to r$ as $n \to \infty$. It can be easily checked that

$$\begin{aligned} D_n(q) &= \left(\frac{1}{4}\right)^q \frac{n}{m+n-1}\frac{n-1}{m+n-2}\cdots\frac{n-q+1}{m+n-q} \\ &\overset{n\to\infty}{\longrightarrow} \left(\frac{1}{4}\right)^q \underbrace{\frac{1}{1+r}\frac{1}{1+r}\cdots\frac{1}{1+r}}_{\text{total of } q \text{ factors}} \\ &= \left(\frac{1}{4(r+1)}\right)^q, \end{aligned}$$

and

$$
\begin{aligned}
|D_n(q)| &= \left(\frac{1}{4}\right)^q \frac{n}{m+n-1}\frac{n-1}{m+n-2}\cdots\frac{n-q+1}{m+n-q} \\
&\leq \left(\frac{1}{4}\right)^q,
\end{aligned}
$$

where $\sum_{q=0}^{\infty}\left(\frac{1}{4}\right)^q = \frac{4}{3} < \infty$. Therefore, by Theorem 3.4.1, the limiting value of $\mu_{n,m}$ is

$$
\begin{aligned}
\mu_r &= \frac{r}{r+1}\left(\sum_{q=0}^{\infty}\left(\frac{1}{4(r+1)}\right)^q - 1\right) \\
&= \frac{r}{r+1}\left(\frac{1}{1-\frac{1}{4(r+1)}} - 1\right) \\
&= \frac{r}{(r+1)(4r+3)},
\end{aligned}
$$

thus,

$$
\mu_{n,m} = \frac{r}{(r+1)(4r+3)} + o(1). \tag{3.4.1}
$$

3.4.2 Limiting Value of $\delta_{n,m}$

As shown at the end of this subsection, the limiting value of $\delta_{n,m}$ is actually 0. However, we need a finer result since $Var(\Gamma_{n,m})$ is expressed in terms of $n \cdot \delta_{n,m}$. In fact, we can prove that $\delta_{n,m} = \delta_r \cdot \frac{1}{n} + o\left(\frac{1}{n}\right)$ as follows.

Recall Equation (3.3.3):

$$
\delta_{n,m} = \sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!}\frac{mn-m^2q}{m+n} - \frac{mn}{(m+n)^2(m+n-1)}.
$$

The summand above multiplied by n becomes

$$\begin{aligned}
& n \cdot \left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!} \frac{mn-m^2q}{m+n} \\
= \;& n \cdot \left(\frac{1}{4}\right)^q \frac{n}{m+n} \frac{n-1}{m+n-1} \cdots \frac{n-q+1}{m+n-q} \frac{1}{m+n-q} \frac{1}{m+n-q-1} \frac{mn-m^2q}{m+n} \\
= \;& \left(\frac{1}{4}\right)^q \frac{n}{m+n} \frac{n-1}{m+n-1} \cdots \frac{n-q+1}{m+n-q} \frac{n}{m+n-q} \frac{n}{m+n-q-1} \frac{mn-m^2q}{n(m+n)},
\end{aligned} \tag{3.4.2}$$

which, as $m/n \to r$, converges to

$$\left(\frac{1}{4}\right)^q \underbrace{\frac{1}{1+r} \frac{1}{1+r} \cdots \frac{1}{1+r}}_{\text{total of } q+2 \text{ factors}} \frac{r-r^2q}{r+1} = \left(\frac{1}{4}\right)^q \left(\frac{1}{r+1}\right)^{q+2} \frac{r-r^2q}{r+1}.$$

Also note that in (3.4.2) each factor after $\left(\frac{1}{4}\right)^q$ is less than 1, so the whole expression (3.4.2) is bounded above by $\left(\frac{1}{4}\right)^q$. Since $\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q = \frac{4}{3} < \infty$, applying Theorem 3.4.1 as in the last subsection, we get

$$\begin{aligned}
& \sum_{q=0}^{\infty} n \cdot \left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n)!} \frac{mn-m^2q}{m+n} \\
\to \;& \sum_{q=0}^{\infty} \left(\frac{1}{4}\right)^q \left(\frac{1}{r+1}\right)^{q+2} \frac{r-r^2q}{r+1} \\
= \;& \frac{12r}{(r+1)(4r+3)^3}.
\end{aligned}$$

Hence, considering $n \cdot \frac{-mn}{(m+n)^2(m+n-1)} \to -\frac{r}{(r+1)^3}$ as $m/n \to r$, we know that $n \cdot \delta_{n,m}$ converges to

$$\delta_r = \frac{12r}{(r+1)(4r+3)^3} - \frac{r}{(r+1)^3} = \frac{r(-4r^2+3)}{(r+1)^3(4r+3)^2}.$$

Thus,

$$\delta_{n,m} = \frac{r(-4r^2+3)}{(r+1)^3(4r+3)^2} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right). \tag{3.4.3}$$

3.4.3 Limiting Value of $\nu_{n,m} - {\mu_{n,m}}^2$

Recall Equation (3.3.4):

$$\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
&= \frac{m^2}{(m+n)^2}\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-2)!}{(n-q)!(m+n-2)!}\left(q+1+\frac{n-(2m+n)q}{m^2+mn-m}\right)\right. \\
&\qquad \left. - \left(\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!}\right)^2\right\} \\
&- \frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}$$

The summand of the first summation above can be written as

$$\left(\frac{1}{4}\right)^q \frac{n}{m+n-2}\,\frac{n-1}{m+n-3}\cdots\frac{n-q+1}{m+n-q-1}\left(q+1+\frac{n-(2m+n)q}{m^2+mn-m}\right).$$

Since $\frac{n}{m+n-2} = \frac{n}{m+n}\left(1+\frac{2}{m+n-2}\right)$, $\frac{n-1}{m+n-3} = \frac{n}{m+n}\left(1+\frac{2n-m}{n(m+n-3)}\right)$, $\cdots$, $\frac{n-q+1}{m+n-q-1} = \frac{n}{m+n}\left(1+\frac{2n-(q-1)m}{n(m+n-q-1)}\right)$, the above becomes

$$\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(1+\frac{2}{m+n-2}\right)\left(1+\frac{2n-m}{n(m+n-3)}\right)\cdots\left(1+\frac{2n-(q-1)m}{n(m+n-q-1)}\right)\left(q+1+\frac{n-(2m+n)q}{m^2+mn-m}\right).$$

By expanding the factors after $\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q$, the above reduces to

$$\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left\{q+1+\frac{n-(2m+n)q}{m^2+mn-m}+(q+1)\left(\frac{2}{m+n-2}+\frac{2n-m}{n(m+n-3)}+\cdots+\frac{2n-(q-1)m}{n(m+n-q-1)}\right)+o\left(\frac{1}{n}\right)\right\}. \tag{3.4.4}$$

We now consider the summand of the second summation in Equation (3.3.4). By the same techniques used above, this summand can be simplified as follows.

$$\left(\frac{1}{4}\right)^q\left(\frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!}\right) = \left(\frac{1}{4}\right)^q\left(\frac{n}{m+n-1}\,\frac{n-1}{m+n-2}\cdots\frac{n-q+1}{m+n-q}\right)$$

Since $\frac{n}{m+n-1} = \frac{n}{m+n}\left(1+\frac{1}{m+n-1}\right)$, $\frac{n-1}{m+n-2} = \frac{n}{m+n}\left(1+\frac{n-m}{n(m+n-2)}\right)$, $\cdots$, $\frac{n-q+1}{m+n-q} = \frac{n}{m+n}$ $\cdot\left(1+\frac{n-(q-1)m}{n(m+n-q-1)}\right)$, the above becomes

$$
\begin{aligned}
&\left(\frac{1}{4}\right)^q \frac{n!(m+n-q-1)!}{(n-q)!(m+n-1)!} \\
&= \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q \left(1+\frac{1}{m+n-1}\right)\left(1+\frac{n-m}{n(m+n-2)}\right)\cdots\left(1+\frac{n-(q-1)m}{n(m+n-q-1)}\right) \\
&= \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q + \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q \left(\frac{1}{m+n-1}+\frac{n-m}{n(m+n-2)}+\cdots+\frac{n-(q-1)m}{n(m+n-q-1)}\right) + o\left(\frac{1}{n}\right).
\end{aligned}
\tag{3.4.5}
$$

Substituting Expression (3.4.4) for the summand of the first summation in Equation (3.3.4), and Expression (3.4.5) for the summand of the second summation in Equation (3.3.4), we get

$$
\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
&= \frac{m^2}{(m+n)^2}\Bigg[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\Bigg\{q+1+\frac{n-(2m+n)q}{m^2+mn-m}+(q+1)\left(\frac{2}{m+n-2}+\frac{2n-m}{n(m+n-3)}+\cdots+\frac{2n-(q-1)m}{n(m+n-q-1)}\right)+o\left(\frac{1}{n}\right)\Bigg\} \\
&\qquad - \Bigg\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p + \sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{1}{m+n-1}+\frac{n-m}{n(m+n-2)}+\cdots+\frac{n-(q-1)m}{n(m+n-q-1)}\right)+o\left(\frac{1}{n}\right)\Bigg\}^2\Bigg] \\
&\quad - \frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}
$$

Note that we have changed the index in the second summation from q to p. Breaking the first summation into two parts, and expanding the quadratic term, we get

$$
\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
=&\frac{m^2}{(m+n)^2}\Bigg[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q(q+1) \\
&\quad+\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left\{\frac{n-(2m+n)q}{m^2+mn-m}+(q+1)\left(\frac{2}{m+n-2}+\frac{2n-m}{n(m+n-3)}+\cdots+\frac{2n-(q-1)m}{n(m+n-q-1)}\right)\right\} \\
&\quad-\left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}^2 \\
&\quad-2\left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}\cdot\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{1}{m+n-1}+\frac{n-m}{n(m+n-2)}+\cdots+\frac{n-(q-1)m}{n(m+n-q-1)}\right) \\
&\quad-\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{1}{m+n-1}+\frac{n-m}{n(m+n-2)}+\cdots+\frac{n-(q-1)m}{n(m+n-q-1)}\right)\right\}^2+o\left(\frac{1}{n}\right)\Bigg] \\
&-\frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}
$$

Combining the summation in the first line with the summation in the third line, and combining the summation in the second line with the summation in the fourth line, it follows that

$$
\begin{aligned}
&\nu_{n,m} - {\mu_{n,m}}^2 \\
=&\frac{m^2}{(m+n)^2}\Bigg[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q(q+1)-\left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}^2 \\
&\quad+\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\Bigg\{\frac{n-(2m+n)q}{m^2+mn-m}+(q+1)\left(\frac{2}{m+n-2}+\frac{2n-m}{n(m+n-3)}+\cdots+\frac{2n-(q-1)m}{n(m+n-q-1)}\right) \\
&\qquad-2\left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}\cdot\left(\frac{1}{m+n-1}+\frac{n-m}{n(m+n-2)}+\cdots+\frac{n-(q-1)m}{n(m+n-q-1)}\right)\Bigg\} \\
&\quad-\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{1}{m+n-1}+\frac{n-m}{n(m+n-2)}+\cdots+\frac{n-(q-1)m}{n(m+n-q-1)}\right)\right\}^2+o\left(\frac{1}{n}\right)\Bigg] \\
&-\frac{mn}{(m+n)^2(m+n-1)}.
\end{aligned}
$$

Distributing the factor $\frac{m^2}{(m+n)^2}$ to each term inside the square bracket, and multiplying the whole equation by n, we obtain

$$
\begin{aligned}
&n(\nu_{n,m} - \mu_{n,m}{}^2) \\
=&\frac{m^2}{(m+n)^2} \cdot n\left[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q(q+1) - \left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}^2\right] \\
&\frac{m^2}{(m+n)^2} \cdot\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left\{\frac{n^2-(2m+n)nq}{m^2+mn-m}+(q+1)\left(\frac{2n}{m+n-2}+\frac{n(2n-m)}{n(m+n-3)}+\cdots+\frac{n(2n-(q-1)m)}{n(m+n-q-1)}\right)\right. \\
&\qquad\left.-2\left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}\cdot\left(\frac{n}{m+n-1}+\frac{n(n-m)}{n(m+n-2)}+\cdots+\frac{n(n-(q-1)m)}{n(m+n-q-1)}\right)\right\} \\
&-\frac{m^2}{(m+n)^2} \cdot \frac{1}{n}\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{n}{m+n-1}+\frac{n(n-m)}{n(m+n-2)}+\cdots+\frac{n(n-(q-1)m)}{n(m+n-q-1)}\right)\right\}^2 \\
&-\frac{mn^2}{(m+n)^2(m+n-1)} + o(1). \qquad (3.4.6)
\end{aligned}
$$

As shown later, the first line at the RHS of the equation above converges to 0 as $m/n \to r$, i.e.,

$$n\left[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q(q+1) - \left(\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right)^2\right] \to 0. \qquad (3.4.7)$$

Meanwhile, we can check that, as $m/n \to r$, the summand in the second and third line at the RHS of Equation (3.4.6) converges to

$$
\begin{aligned}
&\left(\frac{1}{4}\right)^q\left(\frac{1}{r+1}\right)^q\left\{\frac{1-(2r+1)q}{r(r+1)} + (q+1)\left(\frac{2}{r+1}+\frac{2-r}{r+1}+\cdots+\frac{2-(q-1)r}{r+1}\right)\right. \\
&\qquad\left.-2\frac{1}{1-\frac{1}{4(r+1)}}\left(\frac{1}{r+1}+\frac{1-r}{r+1}+\cdots+\frac{1-(q-1)r}{r+1}\right)\right\},
\end{aligned}
$$

hence by applying Theorem 3.4.1 as before, the term in the second and third line at the RHS of Equation (3.4.6) converges to

$$
\begin{aligned}
&\left(\frac{r}{r+1}\right)^2\sum_{q=0}^{\infty}\left(\frac{1}{4}\right)^q\left(\frac{1}{r+1}\right)^q\left\{\frac{1-(2r+1)q}{r(r+1)}+\frac{(q+1)(2q-q(q-1)r/2)}{r+1}-2\frac{4(r+1)}{4r+3}\frac{q-q(q-1)r/2)}{r+1}\right\} \\
=&\left(\frac{r}{r+1}\right)^2\sum_{q=0}^{\infty}\left(\frac{1}{4(r+1)}\right)^q\left\{\frac{-r}{2(r+1)}q^3+\left(\frac{2}{r+1}+\frac{4r}{4r+3}\right)q^2+\left(\frac{-(2r+1)}{r(r+1)}+\frac{r+4}{2(r+1)}-\frac{4(r+2)}{4r+3}\right)q+\frac{1}{r(r+1)}\right\}.
\end{aligned}
$$

Let $x = \frac{1}{4(r+1)}$. Note that $0 < x < 1$ since $r > 0$. Recall that for any $x \in (0,1)$, the following identities hold: $\sum_{q=0}^{\infty} x^q = \frac{1}{1-x}$, $\sum_{q=0}^{\infty} x^q q = \frac{x}{(1-x)^2}$, $\sum_{q=0}^{\infty} x^q q^2 = \frac{x(1+x)}{(1-x)^3}$ and $\sum_{q=0}^{\infty} x^q q^3 = \frac{x(x^2+4x+1)}{(1-x)^4}$. Hence, the limiting value above is further reduced to

$$
\begin{aligned}
&\left(\frac{r}{r+1}\right)^2\left\{\frac{-r}{2(r+1)}\frac{x(x^2+4x+1)}{(1-x)^4}+\left(\frac{2}{r+1}+\frac{4r}{4r+3}\right)\frac{x(1+x)}{(1-x)^3}+\left(\frac{-(2r+1)}{r(r+1)}+\frac{r+4}{2(r+1)}-\frac{4(r+2)}{4r+3}\right)\frac{x}{(1-x)^2}+\frac{1}{r(r+1)}\frac{1}{1-x}\right\}\\
&=\left(\frac{r}{r+1}\right)^2\left\{\frac{-r}{2(r+1)}\frac{4(r+1)+64(r+1)^2+64(r+1)^3}{(4r+3)^4}+\frac{4r^2+12r+6}{(r+1)(4r+3)}\frac{4(r+1)(4r+5)}{(4r+3)^3}\right.\\
&\qquad\left.+\frac{-4r^3-21r^2-24r-6}{2r(r+1)(4r+3)}\frac{4(r+1)}{(4r+3)^2}+\frac{1}{r(r+1)}\frac{4(r+1)}{4r+3}\right\}\\
&=\left(\frac{r}{r+1}\right)^2\left\{\frac{-r(32r^2+96r+66)}{(4r+3)^4}+\frac{4(4r^2+12r+6)(4r+5)}{(4r+3)^4}+\frac{2(-4r^3-21r^2-24r-6)}{r(4r+3)^3}+\frac{4}{r(4r+3)}\right\}\\
&=\left(\frac{r}{r+1}\right)^2\left\{\frac{-32r^4-96r^3-66r^2}{r(4r+3)^4}+\frac{64r^4+272r^3+336r^2+120r}{r(4r+3)^4}+\frac{-32r^4-192r^3-318r^2-192r-36}{r(4r+3)^4}+\frac{256r^3+576r^2+432r+108}{r(4r+3)^4}\right\}\\
&=\left(\frac{r}{r+1}\right)^2\frac{240r^3+528r^2+360r+72}{r(4r+3)^4}\\
&=\left(\frac{r}{r+1}\right)^2\frac{8(r+1)(30r^2+36r+9)}{r(4r+3)^4}.
\end{aligned}
\tag{3.4.8}
$$

Similarly, we can prove that, as $m/n \to r$, the summation in the fourth line at the RHS of Equation (3.4.6) converges to a constant $C > 0$, i.e.,

$$
\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{n}{m+n-1}+\frac{n-m}{m+n-2}+\cdots+\frac{n-(q-1)m}{m+n-q-1}\right)\to C.
$$

Hence

$$
\frac{m^2}{(m+n)^2}\cdot\frac{1}{n}\left\{\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q\left(\frac{n}{m+n-1}+\frac{n-m}{m+n-2}+\cdots+\frac{n-(q-1)m}{m+n-q-1}\right)\right\}^2\to 0
\tag{3.4.9}
$$

Applying the limiting expressions (3.4.7), (3.4.8) and (3.4.9) to Equation (3.4.6), and considering $\frac{mn^2}{(m+n)^2(m+n-1)} \to \frac{r}{(r+1)^3}$ as $m/n \to r$, we conclude that $n(\nu_{n,m} - {\mu_{n,m}}^2)$

converges to

$$
\begin{aligned}
\nu_r &= \left(\frac{r}{r+1}\right)^2 \frac{8(r+1)(30r^2+36r+9)}{r(4r+3)^4} - \frac{r}{(r+1)^3} \\
&= \frac{8r(r+1)^2(30r^2+36r+9) - r(4r+3)^4}{(r+1)^3(4r+3)^4} \\
&= \frac{r(-16r^4+24r^2-9)}{(r+1)^3(4r+3)^4} \\
&= -\frac{r(4r^2-3)^2}{(r+1)^3(4r+3)^4}.
\end{aligned}
$$

Hence,

$$
\nu_{n,m} - {\mu_{n,m}}^2 = -\frac{r(4r^2-3)^2}{(r+1)^3(4r+3)^4} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right). \tag{3.4.10}
$$

We now go back to prove that Expression (3.4.7) is indeed true.

Proof of Formula (3.4.7). Given $\delta > 0$, for n sufficiently large, $0 < r - \epsilon < m/n < r + \epsilon$. Then the first series inside the square brackets in (3.4.7) can be bounded as follows:

$$
\begin{aligned}
& \sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q (q+1) \\
= & \sum_{q=0}^{\infty} \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q (q+1) - \sum_{q=n+1}^{\infty} \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q (q+1) \\
= & \left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2 - \sum_{q=n+1}^{\infty} \left(\frac{n}{4(m+n)}\right)^q (q+1) \\
\leq & \left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2 - \left(\frac{n}{4(m+n)}\right)^{n+1} (n+1+1).
\end{aligned}
$$

When n is sufficiently large, $m/n < r + \epsilon$, so the above reduces to

$$
\sum_{q=0}^{n} \left(\frac{1}{4}\right)^q \left(\frac{n}{m+n}\right)^q (q+1) \leq \left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2 - \left(\frac{1}{4(r+\epsilon+1)}\right)^{n+1} (n+2).
$$

On the other hand, the second series inside the square brackets in (3.4.7) can be bounded as follows:

$$\begin{aligned}
&\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p \\
=\ &\sum_{p=0}^{\infty}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p-\sum_{p=n+1}^{\infty}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p \\
=\ &\frac{1}{1-\frac{n}{4(m+n)}}-\frac{\left(\frac{n}{4(m+n)}\right)^{n+1}}{1-\frac{n}{4(m+n)}} \\
\geq\ &\frac{1}{1-\frac{n}{4(m+n)}}\left(1-\left(\frac{1}{4(r-\epsilon+1)}\right)^{n+1}\right).
\end{aligned}$$

Applying the last two bounds to the LHS of (3.4.7), we get

$$\begin{aligned}
&n\cdot\left[\sum_{q=0}^{n}\left(\frac{1}{4}\right)^q\left(\frac{n}{m+n}\right)^q(q+1)-\left\{\sum_{p=0}^{n}\left(\frac{1}{4}\right)^p\left(\frac{n}{m+n}\right)^p\right\}^2\right] \\
\leq\ &n\cdot\left[\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2-(n+2)\left(\frac{1}{4(r+\epsilon+1)}\right)^{n+1}-\left\{\frac{1}{1-\frac{n}{4(m+n)}}\left(1-\left(\frac{1}{4(r-\epsilon+1)}\right)^{n+1}\right)\right\}^2\right].
\end{aligned}$$

By expanding the quadratic term in the brackets, the RHS of the inequality above can be simplified as

$$\begin{aligned}
&n\cdot\left[\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2-(n+2)\left(\frac{1}{4(r+\epsilon+1)}\right)^{n+1}\right. \\
&\quad\left.-\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2+2\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{n+1}-\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{2(n+1)}\right] \\
=\ &n\cdot\left[-(n+2)\left(\frac{1}{4(r+\epsilon+1)}\right)^{n+1}+2\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{n+1}-\left(\frac{1}{1-\frac{n}{4(m+n)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{2(n+1)}\right].
\end{aligned}$$

The above is further bounded by

$$\begin{aligned}
&n\cdot\left[-(n+2)\left(\frac{1}{4(r+\epsilon+1)}\right)^{n+1}+2\left(\frac{1}{1-\frac{1}{4(r-\epsilon+1)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{n+1}-\left(\frac{1}{1-\frac{1}{4(r+\epsilon+1)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{2(n+1)}\right] \\
=&-n(n+2)\left(\frac{1}{4(r+\epsilon+1)}\right)^{n+1}+2n\left(\frac{1}{1-\frac{1}{4(r-\epsilon+1)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{n+1}-n\left(\frac{1}{1-\frac{1}{4(r+\epsilon+1)}}\right)^2\left(\frac{1}{4(r-\epsilon+1)}\right)^{2(n+1)}.
\end{aligned}$$

Since $\frac{n^s}{t^n}\to 0$ for any fixed $s>0$ and $t>1$, it follows that the three terms above all converge to 0. Hence, the whole bound converges to 0. □

3.4.4 Summary

Using all the limiting values achieved before, we are now able to prove the following main theorem in this chapter.

Theorem 3.4.2. *If* $F_X = F_Y = U[0,1]$ *and* $m/n \to r, r \in (0,\infty)$*, then for internal components,*

$$\begin{aligned} Var(\alpha_{j,m}) &= var_r + o(1), \\ Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) &= cov_r \cdot \frac{1}{m} + o\left(\frac{1}{m}\right), \end{aligned}$$

and the variance of the domination number is in the order of m*, i.e.,*

$$\frac{Var(\Gamma_{n,m})}{m} \to v(r),$$

where

$$\begin{aligned} var_r &= \frac{144r^3 + 360r^2 + 237r + 20}{9(r+1)^2(4r+3)^2}, \\ cov_r &= -\frac{r^2(2304r^4 + 9984r^3 + 16096r^2 + 11440r + 3025)}{9(r+1)^3(4r+3)^4}, \\ v(r) &= \frac{1536r^5 + 6848r^4 + 11536r^3 + 8836r^2 + 2793r + 180}{9(r+1)^3(4r+3)^4}. \end{aligned}$$

Proof. Substituting the limiting value of $\mu_{n,m}$ given in Equation (3.4.1) into Lemma 3.2.2 gives

$$Var(\alpha_{j,m}) = \frac{144r^3 + 360r^2 + 237r + 20}{9(r+1)^2(4r+3)^2} + o(1) \text{ for } j \in \{1, \cdots, m-1\}.$$

Substituting the limiting value of $\mu_{n,m}, \delta_{n,m}$ and $\nu_{n,m} - {\mu_{n,m}}^2$ (given in Equations

(3.4.1), (3.4.3) and (3.4.10), respectively) into Lemma 3.2.3 yields

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = -\frac{r^2(-64r^3 + 208r^2 + 312r + 187)}{9(r+1)^3(4r+3)^2} \cdot \frac{1}{m} + o\left(\frac{1}{m}\right)$$
$$\text{for } j_1 = 0, j_2 = 1, \cdots, m-1,$$

and

$$Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) = -\frac{r^2(2304r^4 + 9984r^3 + 16096r^2 + 11440r + 3025)}{9(r+1)^3(4r+3)^4} \cdot \frac{1}{m} + o\left(\frac{1}{m}\right)$$
$$\text{for } j_1, j_2 = 1, \cdots, m-1.$$

Recall Equation (3.0.1):

$$\begin{aligned} Var(\Gamma_{n,m}) &= 2Var(\alpha_{0,m}) + (m-1)Var(\alpha_{1,m}) \\ &\quad + 2\; Cov(\alpha_{0,m}, \alpha_{m,m}) + 2(m-1)Cov(\alpha_{0,m}, \alpha_{1,m}) + m(m-1)Cov(\alpha_{1,m}, \alpha_{2,m}). \end{aligned}$$

From the orders of the variances and covariances obtained above, we can see that only the terms $Var(\alpha_{1,m})$ and $Cov(\alpha_{1,m}, \alpha_{2,m})$ contribute to $\lim_{n\to\infty} \frac{Var(\Gamma_{n,m})}{m}$. Specifically, applying the limiting values of variances and covariances above to Equation (3.0.1) and dividing the equation by m, we obtain

$$\begin{aligned} &\frac{Var(\Gamma_{n,m})}{m} \\ \to\;& \frac{144r^3 + 360r^2 + 237r + 20}{9(r+1)^2(4r+3)^2} - \frac{r^2(2304r^4 + 9984r^3 + 16096r^2 + 11440r + 3025)}{9(r+1)^3(4r+3)^4} \\ =\;& \frac{1536r^5 + 6848r^4 + 11536r^3 + 8836r^2 + 2793r + 180}{9(r+1)^3(4r+3)^4}. \quad \square \end{aligned}$$

Remark 3.4.1. The complicated formula above is empirically checked by using Monte Carlo simulations in Section 6.1.1. The result (see Figure 6.2) shows that, when n and

m are sufficiently large, the sample variance is in good agreement with the theoretical limiting variance.

Chapter 4

CLT for the Domination Number in One Dimension

In this chapter, we prove the central limit theorem (CLT) for the domination number in one dimension. An important tool used in the proof is "*negative association*." In Section 4.1, we define negatively associated random variables and their CLT; in Section 4.2, by using characteristic functions and applying this CLT for negatively associated random variables, we establish the CLT for the domination number in one dimension.

4.1 Negative Association

The concept of negatively associated (NA) random variables was introduced and carefully studied by Joag-Dev and Proschan [13]. The law of large numbers for NA random variables was established by Taylor *et al.* [14], and the CLT for NA random variables was proven by Newman [15].

Definition 4.1.1. *Consider random variables $X_1, \cdots, X_k$. For every pair of disjoint subsets I, J of $\{1, \cdots, k\}$ and any increasing functions f_I, f_J such that the following covariance exists, if*

$$Cov\{f_I(X_i, i \in I), f_J(X_j, j \in J)\} \leq 0,$$

then $X_1, \cdots, X_k$ are said to be negatively associated (NA).

In this chapter, "NA" may also refer to the vector $\mathbf{X} = (X_1, \cdots, X_k)$ or to the underlying distribution of $\mathbf{X}$.

The following propositions easily follow from the definition above:

Proposition 4.1.1. *A subset of two or more NA random variables is NA.*

Proposition 4.1.2. *A set of independent random variables is NA.*

Proposition 4.1.3. *Random variables defined as increasing functions on disjoint subsets of a set of NA random variables are NA.*

Joag-Dev and Proschan [13] have proven several distributions to be NA, particularly the following:

Proposition 4.1.4. *A multinomial distribution is NA.*

Proposition 4.1.5. *If $X_1, \cdots, X_m$ are m independent random variables with log-concave densities, then the joint conditional distribution of $X_1, \cdots, X_m$ given $\sum_{i=1}^{m} X_i$ is NA.*

Occasionally, as illustrated below, a dependence condition weaker than NA may be used.

Definition 4.1.2. *Random variables $X_1, \cdots, X_k$ are said to be negatively dependent (ND) if for all real $x_1, \cdots, x_k$,*

$$P(X_i > x_i, i = 1, \cdots, k) \leq \prod_{i=1}^{k} P(X_i > x_i)$$

and

$$P(X_i \leq x_i, i = 1, \cdots, k) \leq \prod_{i=1}^{k} P(X_i \leq x_i).$$

Note: It has been shown that that negative association implies negative dependence [13].

Taylor, Patterson and Bozorgnia proved that the SLLN holds for ND random variables [14]. However, here we quote only that part of their theorem to be used in this chapter.

Theorem 4.1.1. *Let $\{X_{k,m} : 1 \leq k \leq m, m \geq 1\}$ be row-wise ND random variable arrays such that $E[X_{k,m}] = 0$ for each k and m. If $|X_{k,m}| \leq M$, then*

$$\frac{1}{m^{1/p}} \sum_{k=1}^{m} X_{k,m} \xrightarrow{a.s.} 0, \quad 0 < p < 2.$$

Newman established the CLT for ND sequences [15]. We now state the distributional limit theorem for row-wise ND random variable arrays, which was proved in [15, Theorem 11].

Theorem 4.1.2. *Suppose $X_{k,m}$ and $Y_{k,m}(1 \leq k \leq m, m \geq 1)$ are triangular arrays such that for each m and k, random variable $X_{k,m}$ is equidistributed with $Y_{k,m}$. Assume for each m, the random variables $X_{k,m}, k = 1, \cdots, m$ are ND, but $Y_{k,m}, k = 1, \cdots, m$ are independent. If in addition,*

$$\lim_{m \to \infty} \frac{\sum_{1 \leq i < j \leq m} Cov(X_{i,m}, X_{j,m})}{m} = 0,$$

then $\frac{1}{m^{1/2}} \sum_{k=0}^{m} X_{k,m}$ converges in distribution to X if and only if $\frac{1}{m^{1/2}} \sum_{k=0}^{m} Y_{k,m}$ converges in distribution to the same X.

When the classical CLT for bounded i.i.d. random variable arrays is applied to $\{Y_{k,m}\}$ in the theorem above, it immediately follows the following theorem.

Theorem 4.1.3. *Let $\{X_{k,m} : 1 \leq k \leq m, m \geq 1\}$ be identically distributed row-wise ND random variable arrays such that $E[X_{k,m}] = 0$ for each k and m. If $|X_{k,m}| \leq M$, and*

$$\lim_{m \to \infty} \frac{\sum_{1 \leq k < l \leq m} Cov(X_{k,m}, X_{l,m})}{m} = 0,$$

then

$$\frac{1}{m^{1/2}} \sum_{k=1}^{m} X_{k,m} \xrightarrow{\mathcal{L}} N(0, \sigma^2),$$

where $\sigma^2 = \lim_{m \to \infty} \frac{Var\left[\sum_{k=1}^{m} X_{k,m}\right]}{m}$.

Remark 4.1.1. Since negative association implies negative dependence, Theorems 4.1.1, 4.1.2 and 4.1.3 are all valid if $\{X_{k,m} : 1 \leq k \leq m, m \geq 1\}$ is row-wise NA.

In our problem, if we could show that $\{\alpha_{j,m}, j = 0, \cdots, m\}$ is NA, then by Theorem 4.1.3, we would immediately obtain the CLT for $\Gamma_{n,m} = \sum_{j=0}^{m} \alpha_{j,m}$. However, we haven't been able to prove the negative association of $\{\alpha_{j,m}, j = 0, \cdots, m\}$. Instead, we first prove that $\{L_{j,m}, j = 0, \cdots, m\}$ is NA, based on which we show that $\{N_{j,m}, j = 0, \cdots, m\}$ is also NA. Next, we express the conditional characteristic function of $\Gamma_{n,m}$ given $N_{j,m}, j = 0, \cdots, m$, in terms of increasing functions of $\{N_{j,m}, j = 0, \cdots, m\}$. Since $\{N_{j,m}, j = 0, \cdots, m\}$ is NA, the increasing functions above are also NA. Finally, applying the SLLN and CLT for the increasing function of $\{N_{j,m}, j = 0, \cdots, m\}$, we show that the expectation of conditional characteristic function converges to a constant, thus the CLT for $\Gamma_{n,m}$ is established. In the rest of this section, we show the negative association of both $\{L_{j,m}, j = 0, \cdots, m\}$ and $\{N_{j,m}, j = 0, \cdots, m\}$.

Lemma 4.1.1. *If $F_X = F_Y = U[0, 1]$, then the random vector $(L_{0,m}, \cdots, L_{m,m})$ is NA.*

Proof. Recall that $L_{j,m} = Y_{(j+1)} - Y_{(j)}$ and suppose that $Z_0, \cdots, Z_m$ are i.i.d. random variables with an exponential distribution, where $\{Z_0, \cdots, Z_m\}$ are independent of $\{L_{0,m}, \cdots, L_{m,m}\}$. Since the exponential distribution is log-concave, from Proposition 4.1.5 we know that given $\sum_{j=0}^{m} Z_i$, the random vector $(Z_0, \cdots, Z_m)$ is NA. Hence by Definition 4.1.1, we know that for any pair of disjoint subsets I, J of $\{0, \cdots, m\}$ and any increasing functions f_I and f_J such that the following covariance exists,

$$Cov\left\{f_I(Z_i, i \in I), f_J(Z_j, j \in J) \,\Big|\, \sum_{k=0}^m Z_k = a\right\} \leq 0, \quad \text{for any } a > 0.$$

Since $f_I\left(\frac{Z_i}{a}, i \in I\right)$ and $f_J\left(\frac{Z_j}{a}, j \in J\right)$ are still increasing functions of $Z_i, i \in I$ and $Z_j, j \in J$, respectively, we have

$$Cov\left\{f_I\left(\frac{Z_i}{a}, i \in I\right), f_J\left(\frac{Z_j}{a}, j \in J\right) \,\Big|\, \sum_{k=0}^m Z_k = a\right\} \leq 0,$$

i.e.,

$$Cov\left\{f_I\left(\frac{Z_i}{\sum_{k=0}^m Z_k}, i \in I\right), f_J\left(\frac{Z_j}{\sum_{k=0}^m Z_k}, j \in J\right) \,\Big|\, \sum_{k=0}^m Z_k = a\right\} \leq 0.$$

Note that the conditional distribution of $\left(\frac{Z_0}{\sum_{i=0}^m Z_i}, \frac{Z_1}{\sum_{i=0}^m Z_i}, \cdots, \frac{Z_m}{\sum_{i=0}^m Z_i}\right)$ given $\sum_{k=0}^m Z_k$ $= a$ is independent of a, so must be the unconditional distribution. Thus, the inequality above yields

$$Cov\left\{f_I\left(\frac{Z_i}{\sum_{k=0}^m Z_k}, i \in I\right), f_J\left(\frac{Z_j}{\sum_{k=0}^m Z_k}, j \in J\right)\right\} \leq 0.$$

Therefore, the random vector

$$\left(\frac{Z_0}{\sum_{i=0}^m Z_i}, \frac{Z_1}{\sum_{i=0}^m Z_i}, \cdots, \frac{Z_m}{\sum_{i=0}^m Z_i}\right)$$

is NA. However, $\left(\frac{Z_0}{\sum_{i=0}^m Z_i}, \frac{Z_1}{\sum_{i=0}^m Z_i}, \cdots, \frac{Z_m}{\sum_{i=0}^m Z_i}\right)$ and $(L_{0,m}, \cdots, L_{m,m})$ have the same distribution, hence $(L_{0,m}, \cdots, L_{m,m})$ is also NA. $\square$

Using Lemma 4.1.1, we show in the following theorem that $(N_{0,m}, \cdots, N_{m,m})$ is NA.

Theorem 4.1.4. *If $F_X = F_Y = U[0,1]$, then the random vector $(N_{0,m}, \cdots, N_{m,m})$ is NA.*

Proof. In Theorem 3.1.1, we have shown that given $L_{k,m} = l_{k,m}, k = 0, \cdots, m$, the random vector $(N_{0,m}, \cdots, N_{m,m})$ is multinomially distributed; hence, it is NA (Proposition 4.1.4). From the definition of negative association, we know that for any disjoint subsets I, J of $\{0, \cdots, m\}$ and increasing functions f_I, f_J, the following inequality holds:

$$
\begin{aligned}
& E\big[f_I(N_{i,m}, i \in I) \cdot f_J(N_{j,m}, j \in J) \mid L_{k,m}, k = 0, \cdots, m\big] \\
\leq\ & E\big[f_I(N_{i,m}, i \in I) \mid L_{k,m}, k = 0, \cdots, m\big] \cdot E\big[f_J(N_{j,m}, j \in J) \mid L_{k,m}, k = 0, \cdots, m\big].
\end{aligned}
$$

Note that given $L_{k,m} = l_{k,m}, k = 0, \cdots, m$, the joint distribution of $\{N_{i,m}, i \in I\}$ only depends on $L_{i,m}, i \in I$, thus $E\big[f_I(N_{i,m}, i \in I) \mid L_{k,m}, k = 0, \cdots, m\big] = E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big]$; similarly, $E\big[f_J(N_{j,m}, j \in J) \mid L_{k,m}, k = 0, \cdots, m\big] = E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]$. Therefore,

$$
\begin{aligned}
& E\big[f_I(N_{i,m}, i \in I) \cdot f_J(N_{j,m}, j \in J) \mid L_{k,m}, k = 0, \cdots, m\big] \\
\leq\ & E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big] \cdot E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big].
\end{aligned}
$$

Taking expectation on both sides of the inequality above yields

$$
\begin{aligned}
& E\Big[E\big[f_I(N_{i,m}, i \in I) \cdot f_J(N_{j,m}, j \in J) \mid L_{k,m}, k = 0, \cdots, m\big]\Big] \\
\leq\ & E\Big[E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big] \cdot E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]\Big].
\end{aligned}
$$

Since Lemma 4.1.1 showed that $(L_{0,m}, \cdots, L_{m,m})$ is NA, and $E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big]$ and $E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]$ are actually increasing functions of

$\{L_{i,m}, i \in I\}$ and $\{L_{j,m}, j \in J\}$, respectively (see Remark 4.1.2 below), applying the definition of NA random vectors yields

$$
\begin{aligned}
& E\Big[E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big] \cdot E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]\Big] \\
\leq\ & E\Big[E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big]\Big] \cdot E\Big[E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]\Big].
\end{aligned}
$$

Combining the two inequalities above produces

$$
\begin{aligned}
& E\Big[E\big[f_I(N_{i,m}, i \in I) \cdot f_J(N_{j,m}, j \in J) \mid L_{k,m}, k = 0, \cdots, m\big]\Big] \\
\leq\ & E\Big[E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big]\Big] \cdot E\Big[E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]\Big],
\end{aligned}
$$

thus

$$
E\big[f_I(N_{i,m}, i \in I) \cdot f_J(N_{j,m}, j \in J)\big] \leq E\big[f_I(N_{i,m}, i \in I)\big] \cdot E[f_J(N_{j,m}, j \in J)\big].
$$

Remark 4.1.2. To finish the proof, we now show $E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m}, i \in I\big]$ and $E\big[f_J(N_{j,m}, j \in J) \mid L_{j,m}, j \in J\big]$ are increasing functions of $\{L_{i,m}, i \in I\}$ and $\{L_{j,m}, j \in J\}$, respectively. By induction, it suffices to show that f_I (or f_J) increases if only one variable increases while other components remain unchanged. That is to show, for any subset $I = \{i_1, \cdots, i_I\}$ of $\{0, \cdots, m\}$, if $l_{i_t} < l'_{i_t}, t \in I$, then

$$
\begin{aligned}
& E\big[f_I(N_{i,m}, i \in I) \mid L_{i,m} = l_i, i \in I\big] \\
\leq\ & E\big[f_I(N_{i,m}, i \in I\}) \mid L_{i,m} = l_i \text{ for } i \in I - \{i_t\}, L_{i_t,m} = l'_{i_t}\big]. \qquad (4.1.1)
\end{aligned}
$$

As illustrated in Figure 4.1, suppose n X-points are independently uniformly distributed in $[0, 1]$, and denote $N_{i,m}, i \in I$ as the number of X-points falling in the interval with length $L_{i,m} = l_i$. Without loss of generality, the intervals have been ordered so

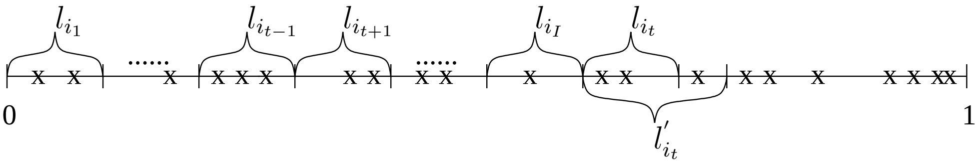

Figure 4.1: A coupling argument with respect to Inequality (4.1.1)

that l_{i_1} corresponds to the leftmost one and l_{i_t} corresponding to the rightmost one, as shown in Figure 4.1. If the length l_{i_t} increases to l'_{i_t}, then $N_{i_t,m}$ will not decrease (possibly increase). This means that when $\{L_{i,m} = l_i \text{ for } i \in I - \{i_t\}, L_{i_t,m} = l'_{i_t}\}$, the random variable $N_{j,m}$ is stochastically larger than the original one when $L_{i,m} = l_i, i \in I$. Since f_I is an increasing function of $N_{i,m}, i \in I$, it follows that Inequality (4.1.1) indeed holds. $\square$

4.2 CLT for the Domination Number

The following is our main theorem in this chapter.

Theorem 4.2.1. *If $F_X = F_Y = U[0,1]$, and $m/n \to r$, then*

$$\frac{1}{m^{1/2}}\big(\Gamma_{n,m} - E[\Gamma_{n,m}]\big) \xrightarrow{\mathcal{L}} N(0, \sigma^2),$$

where $\sigma^2 = \lim\limits_{m\to\infty} \frac{Var[\Gamma n,m]}{m}$. (The exact limiting value was given in Theorem 3.4.2.)

Proof. We define $\mathcal{F}_m = \sigma(N_{0,m}, \cdots, N_{m,m})$ as the σ-field generated by $N_{j,m}, j = 0, \cdots, m$, and let

$$Z_{j,m} = \frac{1}{m^{1/2}}\big(\alpha_{j,m} - E[\alpha_{j,m}]\big)$$

and

$$f_m(t) = E\left[e^{it\sum\limits_{j=0}^{m} Z_{j,m}} \Bigg| \mathcal{F}_m\right].$$

Recall that in Lemma 3.1.2 we have proved that, given $\mathcal{F}_m$, $Z_{j,m}, j = 0, \cdots, m$, are conditionally independent, and each $Z_{j,m}$ is independent of $N_{j',m}$ for any $j' \neq j$. Therefore, the equation above becomes

$$\begin{aligned} f_m(t) &= \prod_{j=0}^{m} E\left[e^{itZ_{j,m}} \mid \mathcal{F}_m\right] \\ &= \prod_{j=0}^{m} E\left[e^{itZ_{j,m}} \mid N_{j,m}\right]. \end{aligned}$$

Since the Taylor expansion tells us that

$$e^{iz} = 1 + iz - \frac{1}{2}z^2 + A(z), \quad \text{where } |A(z)| \leq \frac{|z|^3}{6},$$

the conditional characteristic function of $Z_{j,m}$ can be written as

$$\begin{aligned} E\left[e^{itZ_{j,m}} \mid \mathcal{F}_m\right] &= E\left[e^{itZ_{j,m}} \mid N_{j,m}\right] \\ &= 1 + itE\left[Z_{j,m} \mid N_{j,m}\right] - \frac{t^2}{2}E\left[Z_{j,m}^2 \mid N_{j,m}\right] + r_{j,m}^{(1)}, \end{aligned}$$

where

$$r_{j,m}^{(1)} = E\left[A(tZ_{j,m}) \mid N_{j,m}\right] \leq E\left[\frac{|tZ_{j,m}|^3}{6} \mid N_{j,m}\right].$$

by substituting the formula for $E\left[e^{itZ_{j,m}} \mid \mathcal{F}_m\right]$ into the expression of $f_m(t)$, we get

$$\begin{aligned} log f_m(t) &= \sum_{j=0}^{m} log E\left[e^{itZ_{j,m}} \mid \mathcal{F}_m\right] \\ &= \sum_{j=0}^{m} log\Big(1 + itE\left[Z_{j,m} \mid N_{j,m}\right] - \frac{t^2}{2}E\left[Z_{j,m}^2 \mid N_{j,m}\right] + r_{j,m}^{(1)}\Big). \end{aligned}$$

Again, the Taylor expansion gives

$$log(1+\delta) \;=\; \delta - \frac{\delta^2}{2} + r(\delta), \quad \text{where } |r(\delta)| \le \frac{|\delta|^3}{24} \text{ for } |\delta| < 1, \tag{4.2.1}$$

so $logf_m(t)$ can be further written as

$$\begin{aligned} logf_m(t) \;&=\; \sum_{j=0}^{m} \Big(itE[Z_{j,m} \mid N_{j,m}] - \frac{t^2}{2} E\left[Z_{j,m}^2 \mid N_{j,m}\right] + r_{j,m}^{(1)} \\ &\quad -\frac{1}{2}\Big(itE[Z_{j,m} \mid N_{j,m}] - \frac{t^2}{2} E\left[Z_{j,m}^2 \mid N_{j,m}\right] + r_{j,m}^{(1)} \Big)^2 \\ &\quad + r_{j,m}^{(2)} \Big), \\ \text{where } \left| r_{j,m}^{(2)} \right| \;&\le\; \frac{1}{24} \left| itE[Z_{j,m} \mid N_{j,m}] - \frac{t^2}{2} E\left[Z_{j,m}^2 \mid N_{j,m}\right] + r_{j,m}^{(1)} \right|^3. \end{aligned} \tag{4.2.2}$$

Recall that $\left| r_{j,m}^{(1)} \right|$ is bounded by $E\left[\frac{|tZ_{j,m}|^3}{6} \mid N_{j,m}\right]$. Also note that since $|\alpha_{j,m}|$ is bounded by 2, $|Z_{j,m}| = \left| \frac{1}{m^{1/2}} \big(\alpha_{j,m} - E[\alpha_{j,m}] \big) \right|$ is bounded by $\frac{4}{\sqrt{m}}$. Hence,

$$\left| r_{j,m}^{(1)} \right| \;\le\; \frac{|t|^3 4^3}{6 m^{3/2}}.$$

Let $C_1 \equiv \frac{32|t|^3}{3}$. Then the above is equivalent to

$$\left| r_{j,m}^{(1)} \right| \le C_1 \frac{1}{m^{3/2}}. \tag{4.2.3}$$

We now proceed to the quadratic term in Equation (4.2.2). Based on the same rationale used to derive the bound of $\left| r_{j,m}^{(1)} \right|$, we conclude that

$$\left| E[Z_{j,m} \mid N_{j,m}] \right| \;\le\; \frac{4}{m^{1/2}}, \tag{4.2.4}$$

$$\left| E\left[Z_{j,m}^2 \mid N_{j,m}\right] \right| \;\le\; \frac{16}{m}. \tag{4.2.5}$$

These two bounds, together with $\left|r_{j,m}^{(1)}\right| \leq C_1 \frac{1}{m^{3/2}}$, yield the following expansion of the quadratic term in Equation (4.2.2):

$$\frac{t^2}{2} E[Z_{j,m} \mid N_{j,m}]^2 + r_{j,m}^{(3)}, \tag{4.2.6}$$

where

$$\left|r_{j,m}^{(3)}\right| \leq C_3 \frac{1}{m^{3/2}} \quad \text{for some constant } C_3. \tag{4.2.7}$$

All that now remains is to check $\left|r_{j,m}^{(2)}\right|$ in Equation (4.2.2). From the error bound given in Formula (4.2.1) and the three bounds in Inequalities (4.2.3), (4.2.4) and (4.2.5), we know that when m is large enough,

$$\begin{aligned} \left|r_{j,m}^{(2)}\right| &\leq \frac{1}{24}\left|itE[Z_{j,m} \mid N_{j,m}] - \frac{t^2}{2} E\left[Z_{j,m}^2 \mid N_{j,m}\right] + r_{j,m}^{(1)}\right|^3 \\ &\leq C_2 \frac{1}{m^{3/2}} \quad \text{for some constant } C_2. \end{aligned} \tag{4.2.8}$$

By plugging Formula (4.2.6) back into Equation (4.2.2), we get

$$\begin{aligned} log f_m(t) &= \sum_{j=0}^{m} \bigg(itE[Z_{j,m} \mid N_{j,m}] - \frac{t^2}{2} E\left[Z_{j,m}^2 \mid N_{j,m}\right] + \frac{t^2}{2} E[Z_{j,m} \mid N_{j,m}]^2 \\ &\quad + r_{j,m}^{(1)} + r_{j,m}^{(2)} + r_{j,m}^{(3)} \bigg) \\ &= it \sum_{j=0}^{m} E[Z_{j,m} \mid N_{j,m}] - \frac{t^2}{2} \sum_{j=0}^{m} Var[Z_{j,m} \mid N_{j,m}] \\ &\quad + \sum_{j=0}^{m} \left(r_{j,m}^{(1)} + r_{j,m}^{(2)} + r_{j,m}^{(3)} \right), \end{aligned}$$

i.e.,

$$f_m(t) = e^{it \sum_{j=0}^{m} E[Z_{j,m} | N_{j,m}]} \cdot e^{-\frac{t^2}{2} \sum_{j=0}^{m} Var[Z_{j,m} | N_{j,m}]} \cdot e^{\sum_{j=0}^{m} \left(r_{j,m}^{(1)} + r_{j,m}^{(2)} + r_{j,m}^{(3)} \right)}.$$

Taking the expectation from each side of the equation above gives

$$E[f_m(t)] = E\left[e^{it\sum_{j=0}^{m} E[Z_{j,m}|N_{j,m}]} \cdot e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \cdot e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} \right]. \quad (4.2.9)$$

It should be noted that random vector $\{N_{j,m}\}$ is NA (Lemma 4.1.4), and $E[\alpha_{j,m} \mid N_{j,m}] - E[\alpha_{j,m}]$ is an increasing function of $N_{j,m}$. Therefore, by Proposition 4.1.3, the random vector $\big\{E[\alpha_{j,m} \mid N_{j,m}] - E[\alpha_{j,m}]\big\}$ is also NA. Hence, applying the CLT theorem for row-wise NA random variables (Theorem 4.1.3) yields

$$\begin{aligned} \sum_{j=0}^{m} E[Z_{j,m} \mid N_{j,m}] &= \frac{1}{m^{1/2}} \sum_{j=0}^{m} \Big(E[\alpha_{j,m} \mid N_{j,m}] - E[\alpha_{j,m}] \Big) \\ &\xrightarrow{\mathcal{L}} N(0, {\sigma_1}^2), \end{aligned}$$

where

$$\sigma_1{}^2 = \lim_{m\to\infty} Var\Big(E[\alpha_{j,m} \mid N_{j,m}]\Big) + (m-1)Cov(\alpha_{j_1,m}, \alpha_{j_2,m}).$$

Therefore, by the convergence theorem of characteristic function, we get

$$E\left[e^{it\sum_{j=0}^{m} E[Z_{j,m}|N_{j,m}]} \right] \overset{uniformly}{\longrightarrow} e^{-t^2{\sigma_1}^2/2}. \quad (4.2.10)$$

Similarly, we know that the sequence $\big\{Var[\alpha_{j,m} \mid N_{j,m}]\big\}$ is also NA; hence, by the SLLN theorem for row-wise NA random variables (Theorem 4.1.1), we have

$$\begin{aligned} \sum_{j=0}^{m} Var[Z_{j,m} \mid N_{j,m}] &= \frac{\sum_{j=0}^{m} Var[\alpha_{j,m} \mid N_{j,m}]}{m} \\ &\xrightarrow{a.s} {\sigma_2}^2, \end{aligned}$$

where ${\sigma_2}^2 \equiv \lim_{m\to\infty} E\Big[Var[\alpha_{1,m} \mid N_{j,m}]\Big]$. Thus,

$$e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \xrightarrow{a.s} e^{-t^2\sigma_2^2/2}.$$

From the bounds given in (4.2.3), (4.2.7) and (4.2.8), we get

$$e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} \to e^0 = 1.$$

The two formulas above immediately generate

$$e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \cdot e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} \xrightarrow{a.s} e^{-t^2\sigma_2^2/2}. \tag{4.2.11}$$

According to the definition of almost sure convergence, the result above means that for any $\epsilon > 0$, with probability 1, there exists an $M > 1$ such that, when $m \geq M$,

$$\left| e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \cdot e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} - e^{-t^2\sigma_2^2/2} \right| \leq \epsilon.$$

Since the random variable inside the absolute value sign is bounded, it follows that when $m \geq M$,

$$E\left[\left| e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \cdot e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} - e^{-t^2\sigma_2^2/2} \right|\right] \leq \epsilon.$$

Therefore, by Equation (4.2.9), when $m \geq M$,

$$\begin{aligned}
&\left| E[f_m(t)] - E\left[e^{it\sum_{j=0}^{m} E[Z_{j,m}|N_{j,m}]} \cdot e^{-\frac{t^2}{2}{\sigma_2}^2}\right] \right| \\
\leq\; & E\left[\left| e^{it\sum_{j=0}^{m} E[Z_{j,m}|N_{j,m}]} \right| \cdot \left| e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \cdot e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} - e^{-\frac{t^2}{2}{\sigma_2}^2} \right|\right] \\
=\; & E\left[\left| e^{-\frac{t^2}{2}\sum_{j=0}^{m} Var[Z_{j,m}|N_{j,m}]} \cdot e^{\sum_{j=0}^{m}\left(r_{j,m}^{(1)}+r_{j,m}^{(2)}+r_{j,m}^{(3)}\right)} - e^{-\frac{t^2}{2}{\sigma_2}^2} \right|\right] \leq \epsilon.
\end{aligned}$$

Hence, considering Formula (4.2.10), we get

$$E[f_m(t)] \;\to\; e^{-t^2\sigma_1^2/2} \cdot e^{-t^2\sigma_2^2/2} = e^{-t^2\sigma^2/2},$$

where

$$
\begin{aligned}
\sigma^2 &\equiv {\sigma_1}^2 + {\sigma_2}^2 \\
&= \lim_{m\to\infty} Var\Big[E[\alpha_{j,m} \mid N_{j,m}]\Big] + (m-1)Cov(\alpha_{j_1,m}, \alpha_{j_2,m}) \\
&\quad + \lim_{m\to\infty} E\Big[Var[\alpha_{1,m} \mid N_{j,m}]\Big] \\
&= \lim_{m\to\infty} \frac{Var[\Gamma_{n,m}]}{m}.
\end{aligned}
$$

From the definition of $f_m(t)$ and $Z_{j,m}$, the result above is equivalent to

$$
E\left[e^{it\frac{1}{m^{1/2}}\left(\Gamma_{n,m}-E[\Gamma_{n,m}]\right)}\right] \rightarrow e^{-\frac{t^2}{2}\sigma^2}.
$$

Thus, the result follows from the convergence theorem of characteristic functions. □

Chapter 5

Law of Large Numbers for the Domination Number in Higher Dimensions

Extending the previous results for one dimension to higher dimensions requires a different approach, since the exact distribution of the domination number in the latter case is unknown, and the domination number is not additive on regions separated by Y-points as in the one-dimensional case. In this chapter, we develop some limit theorems for the domination number in higher dimensions by using the SLLN for subadditive processes. Section 5.1 introduces subadditive processes and the related limit theory. However, the ordinary domination number of CCCDs generated by Poisson points is not subadditive. Therefore, to enable use of the SLLN for subadditive processes, we define the constrained domination number, which is then proven to be subadditive. In Section 5.2, by directly applying the ergodic theorem for subadditive processes, we establish the SLLN for the constrained domination number in the Poisson case. Then, in Section 5.3, we prove that the same result still holds for the ordinary domination number, and in Section 5.4 the limiting random variable is proved to be a constant. In Section 5.5, by transferring the Poisson points back to the unit square, we show the weak law of large numbers (WLLN) for the domination number in $[0,1]^2$. Finally, based on the same approach applied in the one-dimensional problem, in Section 5.6 we generalize the WLLN to the case in which the densities f_X and f_Y are positive, bounded and continuous on $[0,1]^2$.

5.1 Introduction to Subadditive Processes

Subadditive ergodic theory was one of probability theory's major achievements in the 1960s and 1970s. Its development was initiated by Hammersley and Welsh [16] and most fully realized by Kingman [17], who also provided an extensive discussion of the theory [18]. Whereas some examples do occur in Smythe and Wierman [19], it was Smythe [20], in his introduction and study of higher dimensional subadditive processes, who defined a two-dimensional subadditive process as follows.

Definition 5.1.1. *A process $\{X_{st}, s < t\}$, where $s, t \in \mathbf{N}^2$, is subadditive if*

P1 Whenever $s < t < u$ and $s_2 < u_2 < t_2$, $X_{st} \leq X_{s(t_1,u_2)} + X_{(s_1,t_2)u}$; whenever

$s < t < u$ and $s_1 < u_1 < t_1$, $X_{st} \leq X_{s(u_1,t_2)} + X_{(u_1,s_2)u}$.

P2 The joint distribution of the process $\{X_{(s_1+1,s_2)(t_1+1,t_2)}\}$ and $\{X_{(s_1,s_2+1)(t_1,t_2+1)}\}$ are the same as those of $\{X_{st}\}$.

P3 $\inf_t E\left[\frac{X_{0t}}{|t|}\right] > -\infty$.

$\{X_{st}, s < t\}$ is called a strongly subadditive process if $P1$ is replaced by the following stronger condition:

P1′ Whenever $s < u < t$,

$$X_{st} - X_{(s_1,u_2)t} - X_{(u_1,s_2)t} + X_{ut} \leq X_{su}.$$

Smythe proved a WLLN for two-dimensional subadditive processes [20] and by adding a complicated condition also gave a SLLN for two-dimensional strongly subadditive processes. In 1981, Akcoglu and Krengel obtained a profound SLLN result for multi-parameter subadditive processes under several natural assumptions [21], and it is to their theorem that our proof of the SLLN for the domination number in higher dimensions mainly resorts. Note that in their paper, Akcoglu and Krengel actually proved the SLLN for *superaddtive processes*. Since $\{-X_I : I \in \mathcal{T}\}$ is superadditive if and only if $\{X_I : I \in \mathcal{T}\}$ is subadditive, any definition or theorem in their paper [21] about superadditive processes translates at once into a corresponding result about subadditive processes. Since we will need to use the SLLN for subadditive processes, we now introduce the subadditive version of their notation and results as follows.

Let $d \geq 1$ be a fixed integer and $\mathbf{S} = \mathbf{R}_+^d$ be the additive semi-group of d-dimensional vectors with nonnegative real coordinates. If $a = (a_i)$ and $b = (b_i)$ are two vectors in $\mathbf{S}$, then $[a, b)$ denotes the set $\{u \mid u = (u_i) \in \mathbf{S}, a_i \leq u_i < b_i\}$ and $\mathcal{T}$ denotes the class of sets of this form. Denote $\vec{0}$ and $\vec{e}$ as the vectors with all coordinates equal to 0 and 1, respectively. Let $J_r = [\vec{0}, r\vec{e}), r > 0$.

Definition 5.1.2. *A continuous subadditive process $\{X_I : I \in \mathcal{T}\}$ satisfies the following:*

A1 If $I_1, \cdots, I_n$ are disjoint sets in $\mathcal{T}$, and $I = \cup_{i=1}^n I_i$ is also in $\mathcal{T}$, then $X_I \leq \sum_{i=1}^n X_{I_i}$.

A2 For any $I_1, \cdots, I_n \in \mathcal{T}$, and any $u \in \mathbf{S}$, the joint distributions of $(X_{I_1}, \cdots, X_{I_n})$ and $(X_{u+I_1}, \cdots, X_{u+I_n})$ are the same.

A3 $\inf\left\{ \frac{E[X_I]}{|I|} : I \in \mathcal{T}, |I| > 0 \right\} = \gamma = \gamma(X) > -\infty.$

Here, $\gamma(X)$ is usually referred to as the time constant of the stochastic process $\{X_I\}$.

We let S_1 denote the set of vectors in $\mathbf{S}$ with integer coordinates and, for a real number $t > 0$, we let $S_t = \{tu \mid u \in S_1\}$ and $\mathcal{T}_t = \{[\vec{a}, \vec{b}) \mid a, b \in S_t\}$. If $\{X_I\}$ is defined only on $\mathcal{T}_t$ for some fixed $t > 0$ and satisfies *A1-A3*, then it is called a *discrete subadditive process.* Akcoglu and Krengel [21] proved the following theorem.

Theorem 5.1.1. *If $\{X_I\}$ is a discrete subadditive process on $\mathcal{T}_1$, then $\lim_{n\to\infty} \frac{X_{J_n}}{|J_n|}$ exists a.e., where $J_n = [\vec{0}, n\vec{e})$.*

Because, as Kingman observed, the continuous analogue of Theorem 5.1.1 is false if no further condition is added, Kingman proposed a natural supplementary condition. The following theorem, taken from Akcoglu and Krengel [21], gives a multiparameter generalization of Kingman's result.

Theorem 5.1.2. *Suppose $\{X_I\}$ to be a subadditive process on intervals with rational end points, and let $\Phi = \sup |X_I|$ where the supremum is taken over all intervals with rational end points in $[\vec{0}, \vec{e})$. If $E[\Phi] < \infty$, then $\lim_{r\to\infty} \frac{X_{J_r}}{|J_r|}$ exists a.e., where $J_r = [\vec{0}, r\vec{e}), r$ is rational.*

Remark 5.1.1. Note that the set of rational numbers in Theorem 5.1.2 can be replaced by any other countable dense subset of $\mathbf{R}^+$ [21]. Recall that a stochastic process X is defined to be separable if there is some countable set of coordinates $\{X_i\}$ whose values determine X. Therefore, under any separability condition on the process $\{X_{J_r}\}$, the a.e.-convergence along each fixed dense countable set implies the a.e.-convergence as r ranges through positive real numbers.

5.2 SLLN for the Constrained Domination Number

We consider two independent homogeneous Poisson processes on the whole 2-dimensional space, $\{X_i\}$ and $\{Y_j\}$, with respective rates λ_X and λ_Y. We now consider a slightly different version of the domination number: the *constrained domination* *number*, which is defined as follows.

Definition 5.2.1. *For a rectangle I with boundary D, the constrained covering ball of $X_i \in I$ is defined by*

$$\bar{B}_I(X_i) = \{\omega \in \Omega : d(\omega, X_i) < min\big\{min_{Y_j \in I} d(Y_j, X_i), min_{z \in D} d(z, X_i)\big\}\big\}.$$

The constrained domination number $\bar{\Gamma}_I$ is the minimum number of constrained covering balls needed to cover all X-points in I. Similarly, the (ordinary) covering ball of $X_i \in I$ is defined by

$$B_I(X_i) = \{\omega \in \Omega : d(\omega, X_i) < \{min_{Y_j \in I} d(Y_j, X_i)\}.$$

The (ordinary) domination number Γ_I is the minimum number of ordinary covering balls needed to cover all X-points in I.

By considering the stochastic process $\{\bar{\Gamma}_I : I = [\vec{a}, \vec{b}), \vec{a}, \vec{b}$ are nonnegative rational points$\}$, we prove the following lemma.

Lemma 5.2.1. $\{\bar{\Gamma}_I\}$ *is a subadditive process.*

Proof. We check the three conditions *A1-A3* in Definition 5.1.2 as follows:

- Suppose $\bar{\mathcal{B}}_{I_i}$ is a constrained class cover of the X-points in I_i. When the boundary of I_i is ignored, the constrained covering balls in $\bar{\mathcal{B}}_{I_i}$ will not decrease (and may increase). Hence, no constrained covering ball $\bar{B}_{I_i}(X_j) \in \bar{\mathcal{B}}_{I_i}$ is any bigger than its corresponding new constrained covering ball, which we denote by

$\bar{B}_{\cup I_i}(X_j)$. Therefore, after ignoring the boundary of I_i, the union of the new constrained covering balls $\{\bar{B}_{\cup I_i}(X_j) : \bar{B}_{I_i}(X_j) \in \bar{\mathcal{B}}_{I_i}\}$ still contains all X-points in I_i. As a result, $\cup_{i=1}^n \{\bar{B}_{\cup I_i}(X_j) : \bar{B}_{I_i}(X_j) \in \bar{\mathcal{B}}_{I_i}\}$ contains all X-points in $\cup_{i=1}^n I_i$; thus, $\sum_{i=1}^n \bar{\Gamma}_{I_i} \geq \bar{\Gamma}_{\cup_{i=1}^n I_i}$.

- *A2* is due to the homogeneity property of Poisson processes.
- *A3* is true since $E[\bar{\Gamma}_I] > 0$ for any I, thus $\inf\left\{\frac{E[\bar{\Gamma}_I]}{|I|} : I \in \mathcal{T}, |I| > 0\right\} \geq 0 > -\infty$. □

Applying Theorem 5.1.2 to the process $\{\bar{\Gamma}_{J_r}\}$ gives the following lemma.

Lemma 5.2.2. $\lim_{r\to\infty} \frac{\bar{\Gamma}_{J_r}}{|J_r|}$ *exists a.e, where* $J_r = [\vec{0}, r\vec{e})$, *and* $r > 0$ *is rational.*

Proof. To apply Theorem 5.1.2, we need only check $E[\Phi] < \infty$, where $\Phi = \sup |X_I|$ when the supremum is taken over all intervals with rational end points in J_1. Since $\forall r \leq 1$, it is readily apparent that

$$\bar{\Gamma}_{J_r} \leq N_X(J_r) \leq N_X(J_1).$$

Hence, $\Phi \leq N_X(J_1)$. Taking the expectation yields $E[\Phi] \leq E[N_X(J_1)] = \lambda_X < \infty$. □

5.3 SLLN for the Ordinary Domination Number

In Lemma 5.2.2, we established the convergence result for the constrained domination number generated by Poisson points. In this section, we prove a similar result for the ordinary domination number.

Recall that when the boundary of a rectangle I is ignored, the constrained covering balls of X-points in I will not decrease (and possibly increase), so the constrained domination number $\bar{\Gamma}_I$ might be bigger (but not smaller) than the domination number Γ_I. Consider a rectangle I that is sufficiently large. Intuitively, the covering balls of most X-points, which are located away from the boundary of I, will not be affected by the existence of the boundary because there are Y-points closer. Therefore, the difference between $\bar{\Gamma}_I$ and Γ_I is only caused by the X and Y-points that are near the boundary of I. In the following formal proof, we show that the effect of these points is negligible in the limit, thus $\lim_{n\to\infty} \frac{\Gamma_{J_n}}{|J_n|}$ exists and equals $\lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|}$, where n is an integer.

With $s_n \ll n$ (to be chosen later), we consider $J_n = \left[\vec{0}, n\vec{e}\right)$, $J'_n = \left[s_n\vec{e}, (n - s_n)\vec{e}\right)$, $J''_n = \left[2s_n\vec{e}, (n-2s_n)\vec{e}\right)$, and $J'''_n = \left[(2+\sqrt{2})s_n\vec{e}, \left(n-(2+\sqrt{2})s_n\right)\vec{e}\right)$ as shown in Figure 5.1. We let F_n denote the event in which all constrained covering balls of X-points in J''_n are contained in J_n, and let E_n denote the event in which there exists at least one Y-point in each of the s_n by s_n squares in $J'_n - J''_n$.

The probability of having at least one Y-point in a particular one of those small squares is $1 - e^{-s_n^2\lambda_Y}$, and the number of small squares is less than $\frac{4n}{s_n}$. Therefore, from the independent increments property of Poisson processes, we know that

$$P(E_n) \geq \left(1 - e^{-s_n^2\lambda_Y}\right)^{\frac{4n}{s_n}}.$$

If $E_n \subseteq F_n$, then we could conclude that $P(F_n) \geq (1 - e^{s_n^2\lambda_Y})^{\frac{4n}{s_n}}$. Next, we show

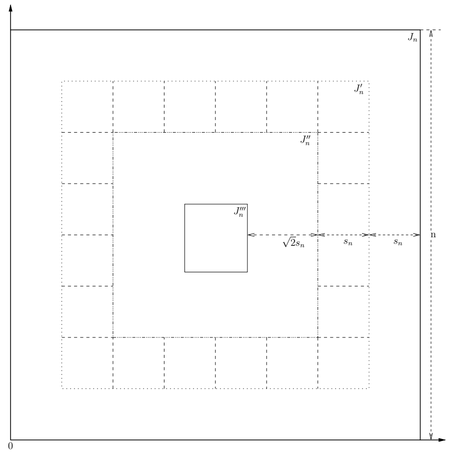

Figure 5.1: The rectangles J_n, J'_n, J''_n and J'''_n

that $E_n \subseteq F_n$ is indeed true. In fact, if there is at least one Y-point in each s_n by s_n square, then the constrained covering ball of any X-point in J''_n cannot go out of J_n. The reason is that for any $X_i \in J''_n$, there is at least one point Y_j in the s_n by s_n square closest to X_i, so the constrained covering ball $\bar{B}(X_i)$ cannot extend very far out of J'_n, hence $\bar{B}(X_i)$ is bounded by J_n. Specifically (but without loss of generality), suppose Y_j is the Y-point closest to X_i, located at the position shown in Figure 5.2. Then the radius of the constrained covering ball $\bar{B}(X_i)$ is $\sqrt{a^2+b^2}$, where the two segments with respective lengths a and b are also shown in Figure 5.2. Considering $a \leq s_n$, we have $\sqrt{a^2+b^2} \leq \sqrt{s_n^2+b^2} \leq b+s_n$. Note that the distance from X_i to the the boundary of J_n is greater or equal than $b+s_n$, thus $\bar{B}(X_i)$ is contained in J_n.

Now we carefully analyze the relation between the constrained domination number $\bar{\Gamma}_{J_n}$ and the ordinary domination number Γ_{J_n}. Let $\Delta_{J_n} = \bar{\Gamma}_{J_n} - \Gamma_{J_n}$. If the boundary

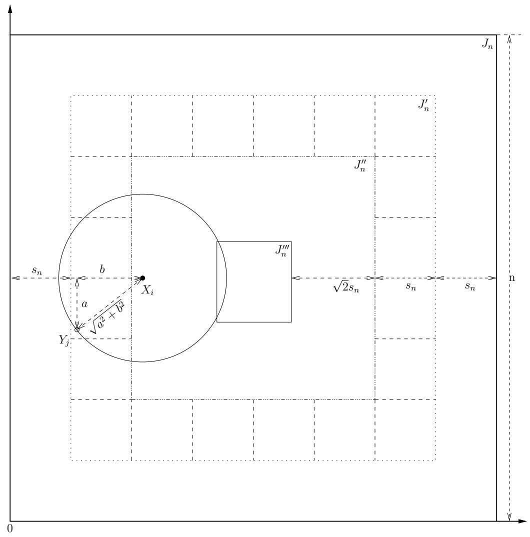

Figure 5.2: An illustration of $E_n \subseteq F_n$

constraint is ignored, the constrained covering balls will not decrease (and might increase) for those X-points whose constrained covering balls touch the boundary; thus, the domination number will not increase, i.e., $\Delta_{J_n} \geq 0$. On the other hand, given the event F_n, the constrained covering ball resizing can only happen for those X-points in $J_n - J''_n$. Although the resized covering balls may cover other X-points in $J_n - J''_n$, the resized balls do not intersect J'''_n. The reason why the resized balls do not intersect J'''_n is that these balls can't reach through the Y-points in the s_n by s_n squares. Specifically (but without loss of generality), suppose Y_j is the Y-point closest to X_i, located at the position shown in Figure 5.3. Then the radius of the resized covering ball $B(X_i)$ is $\sqrt{c^2 + d^2}$, where the two segments with respective lengths c and d are also shown in Figure 5.3. Considering $c \leq s_n$ and $d \leq s_n$, we have $\sqrt{c^2 + d^2} \leq \sqrt{2}s_n$. Note that the distance from X_i to the the boundary of J_n is greater or equal to $\sqrt{2}s_n$, thus $B(X_i)$ does not intersect J_n. Thus, constrained covering ball resizing will decrease $\bar{\Gamma}_n$ by at most the number of X-points in $J_n - J_n'''$, i.e., $\Delta_{J_n} \leq N_X(J_n - J_n''')$.

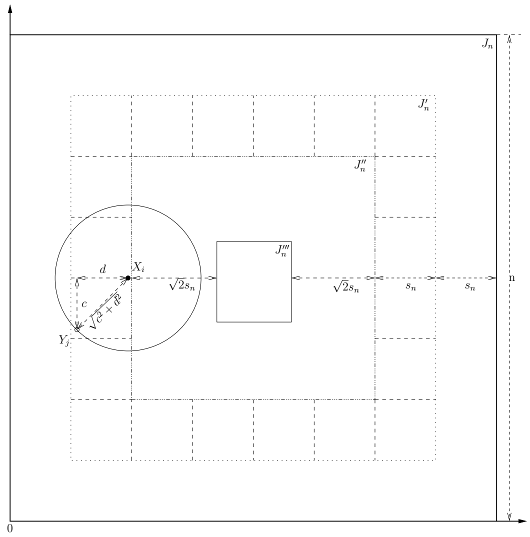

Figure 5.3: An illustration of $\Delta_{J_n} \leq N_X(J_n - J_n''')$

Finally, we estimate $P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon\right)$ as follows:

$$\begin{aligned} P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon\right) &= P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon \mid F_n\right) P(F_n) + P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon \mid F_n^c\right) P(F_n^c) \\ &\leq P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon \mid F_n\right) + P(F_n^c) \\ &\leq E\left[\left(\frac{\Delta_{J_n}}{|J_n|}\right)^2 \mid F_n\right] / \epsilon^2 + P(F_n^c) \quad \text{(by the Markov Inequality).} \end{aligned} \tag{5.3.1}$$

In the first term in the expression above, we have the difficulty that Δ_{J_n} and F_n are dependent. However, we can replace Δ_{J_n} by an upper bound $N_X(J_n - J_n''')$, which is independent of F_n. The reason why $N_X(J_n - J_n''')$ and F_n are independent is that $N_X(J_n - J_n''')$ depends on X-points and F_n depends on Y-points, whereas X-points and Y-points are independent. Therefore, we have

$$\begin{aligned} E\left[\left(\frac{\Delta_{J_n}}{|J_n|}\right)^2 \mid F_n\right]/\epsilon^2 &\leq \frac{E\left[\left(N_X(J_n - J_n''')\right)^2 \mid F_n\right]}{|J_n|^2\epsilon^2} \\ &= \frac{E\left[\left(N_X(J_n - J_n''')\right)^2\right]}{|J_n|^2\epsilon^2}, \end{aligned}$$

Since the Poisson process X has density λ_X, and the second moment equals the variance plus the square of the mean, the above reduces to

$$\begin{aligned} & \frac{\lambda_X|J_n - J_n'''| + \lambda_X^2|J_n - J_n'''|^2}{|J_n|^2\epsilon^2} \\ \leq & \frac{\lambda_X \cdot 4n \cdot 4s_n + \lambda_X^2 \cdot (4n \cdot 4s_n)^2}{\epsilon^2 n^4} \\ \leq & C \cdot \frac{s_n^2}{n^2} \quad \text{for some constant } C. \end{aligned}$$

For the second term of the expression (5.3.1), we get

$$\begin{aligned} P(F_n^c) &= 1 - P(F_n) \\ &\leq 1 - \left(1 - e^{s_n^2\lambda_Y}\right)^{\frac{4n}{s_n}} \\ &= 1 - \left(1 - e^{-s_n^2\lambda_Y}\right)^{e^{s_n^2\lambda_Y}\cdot\frac{4n}{s_n e^{s_n^2\lambda_Y}}}. \end{aligned}$$

Since $\left(1 - \frac{1}{x}\right)^x \uparrow e^{-1}$ as $x \to \infty$, when s_n is sufficiently large, we have

$$\left(1 - e^{-s_n^2\lambda_Y}\right)^{e^{s_n^2\lambda_Y}} > e^{-1-\epsilon}.$$

Therefore, for s_n sufficiently large,

$$P(F_n^c) \leq 1 - e^{-\frac{4(1+\epsilon)n}{s_n e^{s_n^2\lambda_Y}}}.$$

Thus Inequality (5.3.1) becomes

$$P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon\right) \leq C \cdot \frac{s_n^2}{n^2} + \left(1 - e^{-\frac{4(1+\epsilon)n}{s_n e^{s_n^2 \lambda_Y}}}\right).$$

To show that this probability tends to zero, the size of s_n must be chosen carefully. If s_n is chosen to be sufficiently small compared with n, then the first term above goes to zero. If s_n is chosen to be sufficiently large compared with n, then $\frac{n}{s_n e^{s_n^2 \lambda_Y}}$ goes to zero so that the second term in the inequality above also goes to zero. To have both terms converge to zero, we let $s_n = \sqrt{(2+\delta)log(n)/\lambda_Y}$ for some $\delta \in (0,1)$. By Taylor expansion of the main exponential function in the inequality above, we then have

$$\begin{aligned} P\left(\frac{\Delta_{J_n}}{|J_n|} > \epsilon\right) &= O\left(\frac{s_n^2}{n^2} + \frac{n}{s_n e^{s_n^2 \lambda_Y}}\right) \\ &= O\left(\frac{(2+\delta)log(n)/\lambda_Y}{n^2} + \frac{n}{\sqrt{(2+\delta)log(n)/\lambda_Y} \cdot n^{2+\delta}}\right) \\ &= O\left(\frac{1}{n^{1+\delta}}\right). \end{aligned}$$

By the Borel-Contelli lemma, the calculation above immediately implies that $\frac{\Delta_{J_n}}{|J_n|} \xrightarrow{a.s.} 0$. Since $\frac{\Gamma_{J_n}}{|J_n|} = \frac{\bar{\Gamma}_{J_n}}{|J_n|} + \frac{\Delta_{J_n}}{|J_n|}$, and both limits on the right hand side exist a.s., $\lim\limits_{n\to\infty} \frac{\Gamma_{J_n}}{|J_n|}$ exists a.s. and

$$\lim_{n\to\infty} \frac{\Gamma_{J_n}}{|J_n|} = \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \quad a.s.$$

However, our proof of the SLLN for the ordinary domination number is not yet finished, because we need to show that the equation above still holds for Γ_{J_t}, for real t. We first define $\Delta_{J_t} = \bar{\Gamma}_{J_n} - \Gamma_{J_t}$, for any $t = [n, n+1)$. Note that Δ_{J_n} defined before is the difference between the two processes for a single region J_n, whereas Δ_{J_t} defined above is the difference over two different regions, J_t and J_n. While it is possible that $\Gamma_{J_t} > \bar{\Gamma}_{J_n}$, i.e., $\Delta_{J_t} < 0$, Γ_{J_t} can only be larger than $\bar{\Gamma}_{J_n}$ by at most $N_X(J_t - J_n)$, i.e., the number of X-points in $J_t - J_n$. Therefore, we get the following lower bound for Δ_{J_t}:

$$\triangle_{J_t} \geq -N_X(J_t - J_n) \geq -N_X(J_{n+1} - J_n),$$

so

$$\frac{\Delta_{J_t}}{|J_t|} \geq \frac{-N_X(J_{n+1} - J_n)}{|J_t|} \xrightarrow{a.s.} 0 \quad \text{(details shown in the proof of Theorem 5.3.1).}$$

On the other hand, given F_n, the covering balls of X-points in J''_n are completely contained in J_n, so by the same argument for Γ_{J_n}, we know that $\bar{\Gamma}_{J_n}$ can only be larger than Γ_{J_t} by no more than the number of X-points in $J_t - J'''_n$, thus $\frac{\Delta_{J_t}}{|J_t|} \leq \frac{N_X(J_t - J'''_n)}{|J_t|}$. The convergence to zero for the lower and upper bounds of $\frac{\Delta_{J_t}}{|J_t|}$ given above yields the next theorem.

Theorem 5.3.1. $\lim\limits_{t\to\infty} \frac{\Gamma_{J_t}}{|J_t|}$ *exists and is equal to* $\lim\limits_{r\to\infty} \frac{\bar{\Gamma}_{J_r}}{|J_r|}$ *a.s., where t is real and r is rational.*

Proof. For any $t > 0$, there is an integer $n(t)$ s.t. $n(t) \leq t < n(t) + 1$. By the definitions above, $\Gamma_{J_t} = \bar{\Gamma}_{J_{n(t)}} - \triangle_{J_t}$. In addition, we have shown that $\Delta_{J_t} \geq$

$-N_X(J_{n(t)+1} - J_{n(t)})$, so

$$\begin{aligned} \frac{\Gamma_{J_t}}{|J_t|} &\leq \frac{\bar{\Gamma}_{J_{n(t)}} - \Delta_{J_t}}{|J_{n(t)}|} \\ &\leq \frac{\bar{\Gamma}_{J_{n(t)}}}{|J_{n(t)}|} + \frac{N_X(J_{n(t)+1} - J_{n(t)})}{|J_{n(t)}|}. \end{aligned}$$

It should also be noted that

$$\begin{aligned} \frac{N_X(J_{n(t)+1} - J_{n(t)})}{|J_{n(t)}|} &= \frac{N_X(J_{n(t)+1}) - N_X(J_{n(t)})}{|J_{n(t)}|} \\ &= \frac{N_X(J_{n(t)+1})}{|J_{n(t)+1}|} \cdot \frac{|J_{n(t)+1}|}{|J_{n(t)}|} - \frac{N_X(J_{n(t)})}{|J_{n(t)}|} \\ &\xrightarrow{a.s.} \lambda_X \cdot 1 - \lambda_X \quad \text{(the Poisson process } X \text{ has density } \lambda_X) \\ &= 0. \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\Gamma_{J_t}}{|J_t|} \leq \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \quad a.s. \tag{5.3.2}$$

For the other direction, we first write

$$\frac{\Gamma_{J_t}}{|J_t|} = \frac{\Gamma_{J_t}}{|J_t|} \cdot I_{F_{n(t)}} + \frac{\Gamma_{J_t}}{|J_t|} \cdot I_{F^c_{n(t)}}. \tag{5.3.3}$$

Applying the same technique as when we show $\Delta_{J_n} \leq N_X(J_n - J'''_n)$, we know that given $F_{n(t)}$, when the boundary constrain is ignored, the constrained covering ball centered at X-points $\in J''_{n(t)}$ don't change, whereas the covering balls centered at X-points $\in J_t - J''_{n(t)}$ do not intersect with $J'''_{n(t)}$. Therefore, we conclude that $\Delta_{J_t} \leq$

$N_X\left(J_t - J'''_{n(t)}\right)$. Hence, for the first term on the RHS of the equation above, we have

$$\begin{aligned}\frac{\Gamma_{J_t}}{|J_t|} \cdot I_{F_{n(t)}} &\geq \frac{\bar{\Gamma}_{J_{n(t)}} - \Delta_{J_t}}{|J_{n(t)+1}|} \cdot I_{F_{n(t)}} \\ &\geq \left(\frac{\bar{\Gamma}_{J_{n(t)}}}{|J_{n(t)+1}|} - \frac{N_X\left(J_t - J'''_{n(t)}\right)}{|J_{n(t)+1}|}\right) \cdot I_{F_{n(t)}}. \end{aligned} \tag{5.3.4}$$

Recall that we have chosen $s_{n(t)} = \sqrt{(2+\delta)log\big(n(t)\big)/\lambda_Y}$. Because we have shown that $P\left(F^c_{n(t)}\right) = O\left(\frac{n(t)}{s_{n(t)}e^{s^2_{n(t)}\lambda_Y}}\right) = O\left(\frac{1}{n(t)^{1+\delta}}\right)$, the Borel-Contelli lemma gives $I_{F^c_{n(t)}} \xrightarrow{a.s.} 0$. Moreover, we have

$$\lim_{t\to\infty} \frac{\bar{\Gamma}_{J_{n(t)}}}{|J_{n(t)+1}|} = \lim_{t\to\infty} \frac{\bar{\Gamma}_{J_{n(t)}}}{|J_{n(t)}|}\frac{|J_{n(t)}|}{|J_{n(t)} + 1|} = \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \quad a.s.$$

and

$$\begin{aligned}\frac{N_X(J_t - J'''_{n(t)})}{|J_{n(t)+1}|} &= \frac{N_X(J_t)}{|J_{n(t)+1}|} - \frac{N_X(J'''_{n(t)})}{|J_{n(t)+1}|} \\ &= \frac{N_X(J_t)}{|J_t|} \cdot \frac{|J_t|}{|J_{n(t)+1}|} - \frac{N_X(J'''_{n(t)})}{|J'''_{n(t)}|} \cdot \frac{|J'''_{n(t)}|}{|J_{n(t)+1}|} \\ &\xrightarrow{a.s.} \lambda_X \cdot 1 - \lambda_X \cdot 1 = 0.\end{aligned}$$

Thus, substituting the formulas above into Inequality (5.3.4), we immediately get

$$\liminf_{t\to\infty} \frac{\Gamma_{J_t}}{|J_t|} \cdot I_{F_{n(t)}} \geq \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \quad a.s.$$

In addition, note that $\frac{\Gamma_{J_t}}{|J_t|} \leq \frac{N_X(J_t)}{|J_t|} \xrightarrow{a.s.} \lambda_X$. Also, recall that because $I_{F^c_{n(t)}} \xrightarrow{a.s.} 0$,

$$\frac{\Gamma_{J_t}}{|J_t|} \cdot I_{F^c_{n(t)}} \xrightarrow{a.s.} 0.$$

Therefore, we can incorporate the two results above into Equation (5.3.3) to get

$$\liminf_{t\to\infty} \frac{\Gamma_{J_t}}{|J_t|} \geq \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \quad a.s. \tag{5.3.5}$$

Furthermore, based on Inequalities (5.3.2) and (5.3.5), we conclude that

$$\lim_{t\to\infty} \frac{\Gamma_{J_t}}{|J_t|} = \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \quad a.s.$$

Thus, since $\left\{\frac{\bar{\Gamma}_{J_n}}{|J_n|}\right\}$ is a subsequence of $\left\{\frac{\bar{\Gamma}_{J_r}}{|J_r|}\right\}$, it follows that $\lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} = \lim_{r\to\infty} \frac{\bar{\Gamma}_{J_r}}{|J_r|}$ a.s., where the existence of the latter is guaranteed by Lemma 5.2.2. □

5.4 Convergence to a Constant

Theorem 5.1.1 alone does not identify the limiting random variable $\xi = \lim_{I\to\infty} \frac{X_I}{|I|}$ for a subadditive process $\{X_I\}$; however, in his paper [20], Smythe showed that when the subadditive process is independent, the limit is simply the time constant $\gamma(X)$. Next, we first formally give the definition of independent subadditive processes and state the result by Smythe, then describe the idea of his proof.

Definition 5.4.1. *A subadditive process is independent if the random variables* $\{X_{s_i,t_i}\}$ *are independent for disjoint rectangles* $\{R_{s_i,t_i}\}_{i=1,\cdots,n}$.

Theorem 5.4.1. *If a discrete subadditive process* $\{X_I\}$ *is independent, then* $\xi = \lim_{n\to\infty} \frac{X_{J_n}}{|J_n|}$ *is equal to the constant* $\gamma(X)$ *a.s.*

Let $\mathcal{F}$ be the σ-field of events generated by the process $\{X_I\}$ and invariant under both the shifts $\theta_1 : X_{(s_1,s_2),(t_1,t_2)} \to X_{(s_1+1,s_2),(t_1+1,t_2)}$ and $\theta_2 : X_{(s_1,s_2),(t_1,t_2)} \to X_{(s_1,s_2+1),(t_1,t_2+1)}$. Then, as shown by Kingman ([17], page 504), the limit $\xi = \lim_{n\to\infty} \frac{X_{J_n}}{|J_n|}$

can be written in the following form:

$$\xi = \lim_{n\to\infty} \frac{E[X_{J_n} \mid \mathcal{F}]}{|J_n|}, \quad a.s.$$

It is important to note that even though Kingman's original result only addresses the one-dimensional case, this principle has been adapted to prove the same result for higher dimensions [20, page 777].

Since if the invariant σ-field $\mathcal{F}$ is generated by an independent process, then $\mathcal{F}$ is trivial [20, page 782], we immediately get

$$\xi = \lim_{n\to\infty} \frac{E[X_{J_n}]}{|J_n|} = \gamma(X), \quad a.s.$$

In our problem, the subadditive process $\{\bar{\Gamma}_{J_n}\}$ is independent, so $\lim\limits_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} = \lim\limits_{n\to\infty} \frac{E[\bar{\Gamma}_{J_n}]}{|J_n|}$ $= \gamma(\bar{\Gamma})$ a.s. Therefore, by Theorem 5.3.1, we immediately achieve the following result.

Theorem 5.4.2. $\lim\limits_{t\to\infty} \frac{\Gamma_{J_t}}{|J_t|} = \lim\limits_{n\to\infty} \frac{E[\bar{\Gamma}_{J_n}]}{|J_n|} = \gamma(\bar{\Gamma})$ *a.s.*

Remark 5.4.1. In the theorem above, the limiting value is given as $\gamma(\bar{\Gamma})$ instead of $\gamma(\Gamma)$. The reason is that the process $\{\Gamma_{J_n}\}$ is not subadditive, hence $\gamma(\Gamma)$ is not even well-defined.

5.5 WLLN for the Domination Number in $[0,1]^2$ with Uniform Densities

In the previous sections, we have established the SLLN for the domination number generated by Poisson points in $\mathbf{R}^2$. In this section, we transfer the result of the Poisson case to the unit square $[0,1]^2$. Denote $\Gamma'_{n,m}$ as the domination number generated by n X-points and m Y-points, and assume that both the X-points and Y-points are uniformly distributed in $[0,1]^2$. We prove the weak law of large numbers (WLLN) for $\Gamma'_{n,m}$.

Theorem 5.5.1. *If* $\frac{m}{n} \to r, r \in (0,\infty)$*, then* $\lim_{n\to\infty} \frac{\Gamma'_{n,m}}{n} = g(r)$ *in probability, where* $g(r) = \gamma(\bar{\Gamma}) = \gamma(\bar{\Gamma}, r)$*.*

Proof. In the Poisson case, we let the rates be $\lambda_X = 1$ and $\lambda_Y = r$. For any integer $n > 0$, we let $t(n)$ be the smallest real number t such that there are $n+1$ X-points in J_t. Note that the $(n+1)$-st X-point is on the boundary of $J_{t(n)}$, and the other n X-points are in the interior of $J_{t(n)}$. Define $\Gamma_{n,m_n} = \Gamma_{J_{t(n)}}$, where m_n is the random number of Y-points in $J_{t(n)}$. We know by Theorem 5.4.2 that $\lim_{n\to\infty} \frac{\Gamma_{J_{t(n)}}}{|J_{t(n)}|} = \gamma(\bar{\Gamma})$ *a.s*, since $t(n) \to \infty$ *a.s.* as $n \to \infty$. Equivalently,

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_n}}{|J_{t(n)}|} = \gamma(\bar{\Gamma}) \quad a.s.$$

Combining the equation above with the fact that $\lim_{n\to\infty} \frac{n}{|J_{t(n)}|} = \lambda_X = 1$ *a.s.*, we get

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_n}}{n} = \lim_{n\to\infty} \frac{\Gamma_{n,m_n}}{n} \cdot \frac{n}{|J_{t(n)}|} = \gamma(\bar{\Gamma}) \quad a.s.$$

Since almost sure convergence implies convergence in distribution, it follows that

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_n}}{n} = \gamma(\bar{\Gamma}) \quad \text{in distribution.} \tag{5.5.1}$$

From the conditional uniformity property of Poisson processes, the n X-points and m_n Y-points are both uniformly distributed in $J_{t(n)}$. Recall that the desired number of Y-points is $m = m(n)$, which is a non-random function of n. For simplicity, we will use m rather than $m(n)$ in the following proof. On the other hand, m_n is the random number of Y-points in the Poisson case. If $m_n < m$, we add $m - m_n$ Y-points in the region $J_{t(n)}$ in a uniform way. Similarly, if $m_n > m$, then we delete $m_n - m$ Y-points uniformly from the m_n Y-points in $J_{t(n)}$. And if $m_n = m$, no change is needed. After such modification, the original m_n Y-points become m Y-points. Let $\Gamma_{n,m}$ denote the domination number generated by the n X-points and the m Y-points, which are uniformly distributed in $J_{t(n)}$. Note that $\Gamma_{n,m}$ has the same distribution as $\Gamma'_{n,m}$. Hence, if we can prove

$$\lim_{n\to\infty} \frac{\Gamma_{n,m}}{n} = \gamma(\bar{\Gamma}) \quad \text{in distribution,} \tag{5.5.2}$$

then we have $\lim_{n\to\infty} \frac{\Gamma'_{n,m}}{n} = \lim_{n\to\infty} \frac{\Gamma_{n,m}}{n} = \gamma(\bar{\Gamma})$ in distribution, hence $\lim_{n\to\infty} \frac{\Gamma'_{n,m}}{n} = \gamma(\bar{\Gamma})$ in probability, since the limit is a constant. So, the problem reduces to showing Equation (5.5.2). In fact, if we let $\Delta_{n,m_n} = \Gamma_{n,m} - \Gamma_{n,m_n}$, and if we can prove

$$\frac{\Delta_{n,m_n}}{n} \to 0 \quad \text{in probability,}$$

then considering the result (5.5.1), by Slutsky's theorem we get

$$\begin{aligned} \lim_{n\to\infty} \frac{\Gamma_{n,m}}{n} &= \lim_{n\to\infty} \frac{\Gamma_{n,m_n}}{n} + \lim_{n\to\infty} \frac{\Delta_{n,m_n}}{n} \\ &= \gamma(\bar{\Gamma}) \quad \text{in distribution.} \end{aligned}$$

All that remains from the discussion above is to show the following lemma is true. □

Lemma 5.5.1. $\frac{\Delta_{n,m_n}}{n} \to 0$ *in probability.*

Proof. Case 1: Adding Points.

We first consider the case of adding one new Y-point: Y_a, when $m_n - m = -1$. As illustrated in Figure 5.4, if Y_a falls into the covering ball $B(X_i)$ of some point

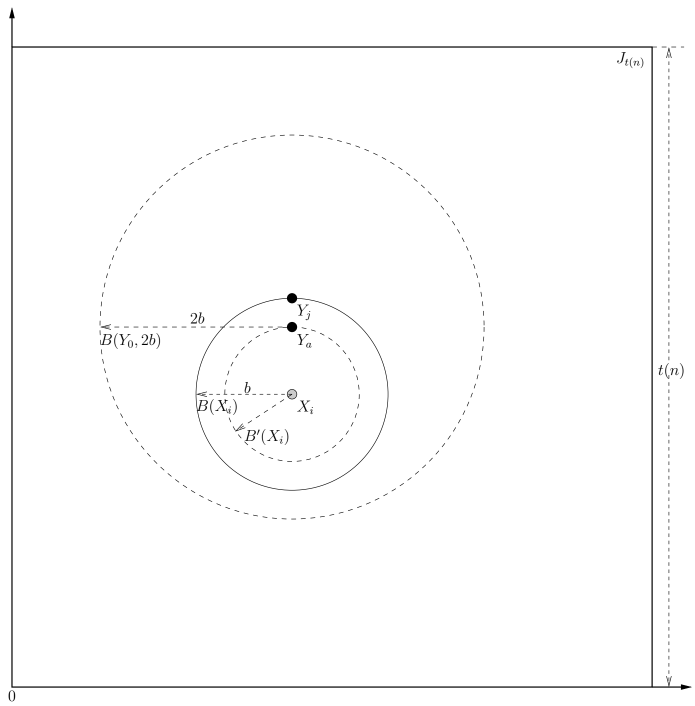

Figure 5.4: The result of adding one new point Y_0

X_i, the covering ball $B(X_i)$ will decrease to $B'(X_i)$ so that the domination number may increase (but never decrease). Such an increase can be at most the number of X-points in $B(X_i)$.

Note that it is possible for Y_a to fall into more than one covering ball. To take this into account, define the random variable $B_a \equiv$ maximum radius of all balls that contain Y_a but contain no Y-points. We know that given $B_a = b > 0$, the covering balls into which Y_a could fall must be contained in the ball $B(Y_a, 2b)$, which is centered at Y_a with radius $2b$. Otherwise, if there exists a covering ball that contains Y_a but isn't contained in $B(Y_a, 2b)$, then that covering ball must have a radius greater than b but contain no Y-points, which contradicts $B_a = b$. Therefore, Δ_{n,m_n} is bounded above by the number of X-points in $B(Y_a, 2b)$, thus

$$0 \leq \Delta_{n,m_n} \leq \sum_{i=1}^{n} I_{\{X_i \in B(Y_a,2b)\}}. \tag{5.5.3}$$

Next, we calculate an upper bound for $P(B_a > b)$. Define the event

$$F(Y_a, b) \equiv \{\exists \text{ a ball in } B(Y_a, 2b) \text{ with radius } b \text{ s.t. there exists no } Y\text{-point in it}\}.$$

Note that in the definition above, the ball is a subset of $B(Y_a, 2b)$, but it is not necessarily centered at a X-point or Y-point. From the definition of B_a, it is easy to see that $\{B_a > b\} \subset F(Y_a, b)$. Next we will find an upper bound for $P\big(F(Y_a, b)\big)$. As shown in Figure 5.5, suppose we equally divide the square centered at Y_a with side length $4b$ into $8^2(= 64)$ smaller squares, and refer to the 64 small balls in the squares with radius $b/4$ as grid balls. If $F(Y_a, b)$ is true, i.e., there exists a ball in $B(Y_a, 2b)$ with radius b such there are no Y-points in it, then that ball must contain a grid ball that covers no Y-point (as illustrated in Figure 5.5). Therefore, if $F(Y_a, b)$ is true, then there must exist a grid ball containing no Y-point. Since the Poisson process Y has density λ_Y, we know the probability that a particular grid ball covers no Y-point

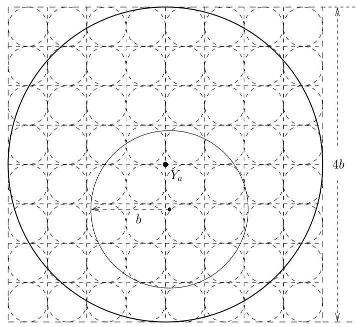

Figure 5.5: The affected region of an added point Y_a

is $e^{-\pi(b/4)^2\lambda_Y}$. Therefore,

$$\begin{aligned} P(B_a > b) &\leq P\big(F(Y_a, b)\big) \\ &\leq 64e^{-\pi r(b/4)^2}. \end{aligned}$$

Applying Formula (5.5.3), we have

$$\begin{aligned} & P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid \{m_n - m = -1\} \cap \{B_a = b\}\right) \\ \leq\ & P\left(\sum_{i=1}^{n} I_{\{X_i \in B(Y_a, 2b)\}} > n\epsilon \mid \{m_n - m = -1\} \cap \{B_a = b\}\right). \end{aligned}$$

Since X-points are independent of Y-points, and all X_i are identically distributed, the RHS of the inequality above reduces to $P\left(\sum_{i=1}^{n} I_{\{X_i \in B(Y_a,2b)\}} > n\epsilon\right)$, which, by the Markov Inequality, is further bounded by $\frac{E\left[\sum_{i=1}^{n} I_{\{X_i \in B(Y_a,2b)\}}\right]}{n\epsilon} = \frac{P(X_i \in B(Y_a,2b))}{\epsilon}$. Note that if $B(Y_a, 2b)$ is contained in $J_{t(n)}$, then $P\left(X_i \in B(Y_a, 2b)\right) = \frac{\pi(2b)^2}{|J_{t(n)}|}$. However, if Y_a is near the boundary of $J_{t(n)}$, then it is possible that only part of $B(Y_a, 2b)$ is contained in $J_{t(n)}$, hence $P\left(X_i \in B(Y_a, 2b)\right) \leq \frac{\pi(2b)^2}{|J_{t(n)}|}$. Summarizing the discussion above, we get

$$P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid \{m_n - m = -1\} \cap \{B_a = b\}\right) \leq \frac{\pi(2b)^2}{|J_{t(n)}| \cdot \epsilon}.$$

Hence, given $t(n)$,

$$\begin{aligned}
& P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = -1\right) \\
= & \int_{b=0}^{\sqrt{2}t(n)} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid \{m_n - m = -1\} \cap \{B_a = b\}\right) dF_{B_a}(b) \\
\leq & \int_{b=0}^{\sqrt{2}t(n)} \frac{\pi(2b)^2}{|J_{t(n)}| \cdot \epsilon} d\big(1 - P(B_a > b)\big) \\
= & \int_{b=0}^{\sqrt{2}t(n)} P(B_a > b) d\left(\frac{\pi(2b)^2}{|J_{t(n)}| \cdot \epsilon}\right).
\end{aligned}$$

Recalling that $P(B_a > b) \leq 64e^{-\pi r(b/4)^2}$, we further bound the above as follows:

$$P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = -1\right) \leq \int_{b=0}^{\sqrt{2}t(n)} 64e^{-\pi r(b/4)^2} d\left(\frac{\pi(2b)^2}{|J_{t(n)}| \cdot \epsilon}\right) \leq \frac{C}{|J_{t(n)}|},$$

where $C > 0$ is a constant. Therefore, without conditioning on $t(n)$, we have

$$P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = -1\right) \leq C \cdot E\left[\frac{1}{|J_{t(n)}|}\right].$$

Next, we consider the case of adding one or more new Y-points: $Y_a^1, \cdots, Y_a^{|m_n - n|}$, when $m_n - m \leq -1$. Similarly, define $B_a^l \equiv$ maximum radius of the covering balls containing $Y_a^l, l = 1, \cdots, |m_n - m|$. Given $m_n - m = \rho_n \in \{-\delta n, \cdots, -1\}$ and $B_a^l = b^l > 0$, by applying the same arguments as above, we have

$$\begin{aligned}
P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid \{m_n - m = \rho_n\} \cap \left\{B_a^l = b^l\right\}\right) & \leq \frac{P\left(X_i \in \cup_{l=1}^{|\rho_n|} B(Y_a^l, 2b^l)\right)}{\epsilon} \\
& \leq \sum_{l=1}^{|\rho_n|} \frac{\pi(2b^l)^2}{|J_{t(n)}| \cdot \epsilon}.
\end{aligned}$$

Using $P(B_a^l > b^l) \leq 64e^{-\pi r(b^l/4)^2}$ as before, and recalling we have chosen $|\rho_n| \leq \delta n$, we can finally get the following bound:

$$
\begin{aligned}
P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) &\leq |\rho_n| \cdot C \cdot E\left[\frac{1}{|J_{t(n)}|}\right] \\
&\leq \delta n \cdot C \cdot E\left[\frac{1}{|J_{t(n)}|}\right]. \qquad (5.5.4)
\end{aligned}
$$

Note that for any $\delta > 0$, by the law of total probability, we have

$$
\begin{aligned}
& P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \\
= & \sum_{\rho_n < -\delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n) \\
& + \sum_{-\delta n \leq \rho_n \leq -1} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n) \\
& + P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = 0\right) P(m_n - m = 0) \\
& + \sum_{1 \leq \rho_n \leq \delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n) \\
& + \sum_{\rho_n > \delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n).
\end{aligned}
$$

Applying $P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) \leq 1$ to the first and last summation above, and applying Inequality (5.5.4) to the second summation above, the equation above can be bounded as follows:

$$\begin{aligned}
P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \leq\ & \sum_{\rho_n < -\delta n} P(m_n - m = \rho_n) \\
& + \sum_{-\delta n \leq \rho_n \leq -1} \delta n \cdot C \cdot E\left[\frac{1}{|J_{t(n)}|}\right] P(m_n - m = \rho_n) \\
& + P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = 0\right) P(m_n - m = 0) \\
& + \sum_{1 \leq \rho_n \leq \delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n) \\
& + \sum_{\rho_n > \delta n} P(m_n - m = \rho_n)\,.
\end{aligned}$$

Combining the first and last summations above, and factoring out the term $\delta n \cdot C \cdot E\left[\frac{1}{|J_{t(n)}|}\right]$ in the second summation above, we have

$$\begin{aligned}
P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \leq\ & P(|m_n - m| > \delta n) \\
& + \delta n \cdot C \cdot E\left[\frac{1}{|J_{t(n)}|}\right] \sum_{-\delta n \leq \rho_n \leq -1} P(m_n - m = \rho_n) \\
& + P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = 0\right) P(m_n - m = 0) \\
& + \sum_{1 \leq \rho_n \leq \delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n)\,.
\end{aligned}$$

Note that the summation in the second term above equals $P(-\delta n \leq m_n - m \leq -1)$, which is further bounded by 1. In addition, if $m_n - m = 0$, then $\Delta_{n,m_n} = 0$, thus $P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = 0\right) = 0$, hence the third term above equals 0. Therefore, the inequality above reduces to

$$
\begin{aligned}
P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \; & \leq P(|m_n - m| > \delta n) \\
& + \delta n \cdot C \cdot E\left[\frac{1}{|J_{t(n)}|}\right] \\
& + \sum_{1 \leq \rho_n \leq \delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n)\,.
\end{aligned}
$$

Since $E\left[\frac{n}{|J_{t(n)}|}\right]$ is bounded, the inequality above yields

$$
\begin{aligned}
P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \; & \leq \; P(|m_n - m| > \delta n) \\
& + \delta C_1 \\
& + \sum_{1 \leq \rho_n \leq \delta n} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) P(m_n - m = \rho_n) \\
& \qquad \text{for some constant } C_1 > 0, \qquad (5.5.5)
\end{aligned}
$$

Case 2: Deleting Points.

In contrast to adding a point, deleting an existing point Y_d can only decrease the domination number or leave it unchanged. As illustrated in Figure 5.6, if Y_d is on the boundary of $B(X_i)$ of some X_i, then deleting Y_d will cause $B(X_i)$ to increase to $B'(X_i)$, which we refer to as the enlarged covering ball. The enlarged covering ball $B'(X_i)$ has a radius equal to the distance between X_i and the second nearest Y-point: Y_j. It is worth noting that the domination number can decrease by at most the number of X-points in $B'(X_i)$.

It is also possible for Y_d to fall into more than one enlarged covering ball. Refer to the original Y-points except Y_d as Y'-points. Define the random variable $B_d \equiv$

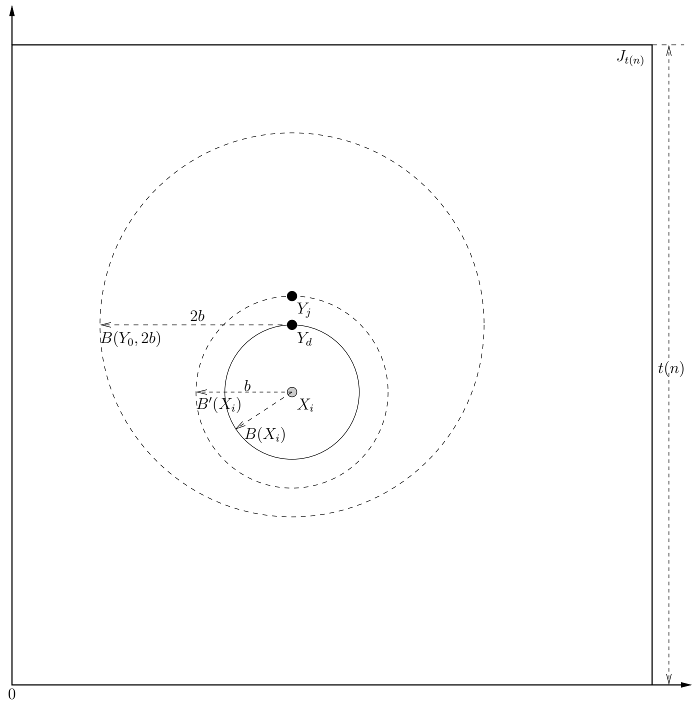

Figure 5.6: The result of deleting one existing point Y_d

maximum radius of all balls that contain Y_d but contain no Y'-points. Given $B_d = b > 0$, the enlarged covering balls into which Y_d could fall must be contained in the ball $B(Y_d, 2b)$. Otherwise, if there exists an enlarged covering ball that contains Y_d but is not contained in $B(Y_d, 2b)$, then that enlarged covering ball must have a radius greater than b but contain no Y'-point, which contradicts $B_d = b$. Therefore, $|\Delta_{n,m_n}|$ is bounded above by the number of X-points in $B(Y_d, 2b)$, thus

$$0 \geq \Delta_{n,m_n} \geq -\sum_{i=1}^{n} I_{\{X_i \in B(Y_d, 2b)\}}.$$

Define the event

$$F'(Y_d, b) \equiv \{\exists \text{ a ball in } B(Y_d, 2b) \text{ with radius } b \text{ s.t. there exists no } Y'\text{-point in it}\}.$$

As in the the case of adding a point, it is easy to see that $\{B_d > b\} \subset F'(Y_d, b)$. Hence, conditioning on $t(n)$, we can get the following upper bound of $P(B_d > b)$:

$$\begin{aligned} P(B_d > b) &\leq P\left(F'(Y_d, b)\right) \\ &\leq 64 \cdot P(\text{a grid ball contains no } Y'\text{-point}) \\ &= 64\left(1 - \frac{\pi(b/4)^2}{t(n)^2}\right)^m . \end{aligned}$$

Therefore, using the same argument as in the case of adding points, for any $\rho_n \in \{1, \cdots, \delta n\}$, we get

$$\begin{aligned} & P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) \\ \leq\ & \delta n \cdot E\left[\int_{b=0}^{\sqrt{2}t(n)} 64\left(1 - \frac{\pi(b/4)^2}{t(n)^2}\right)^m d\left(\frac{\pi(2b)^2}{|J_{t(n)}| \cdot \epsilon}\right)\right]. \end{aligned} \tag{5.5.6}$$

By applying $|J_{t(n)}| = t(n)^2$ and adjusting the factors in $d\left(\frac{\pi(2b)^2}{|J_{t(n)}|\cdot\epsilon}\right)$, the integral above can be further calculated as follows:

$$\begin{aligned} & \int_{b=0}^{\sqrt{2}t(n)} 64\left(1 - \frac{\pi(b/4)^2}{t(n)^2}\right)^m d\left(\frac{\pi(2b)^2}{|J_{t(n)}| \cdot \epsilon}\right) \\ =\ & \int_{b=0}^{\sqrt{2}t(n)} \frac{64^2}{\epsilon}\left(1 - \frac{\pi(b/4)^2}{t(n)^2}\right)^m d\left(\frac{\pi(b/4)^2}{t(n)^2}\right) \\ =\ & \frac{64^2}{\epsilon}\left[\frac{1}{m+1}\left(1 - \frac{\pi(b/4)^2}{t(n)^2}\right)^{m+1}\right]_{b=\sqrt{2}t(n)}^{b=0} \\ =\ & \frac{64^2}{\epsilon}\frac{1}{m+1}\left(1 - \left(1 - \frac{\pi}{32}\right)^{m+1}\right) \leq \frac{C'}{m} \quad \text{for some constant } C' > 0. \end{aligned}$$

Hence, for any $\rho_n \in \{1, \cdots, \delta n\}$, when n is sufficiently large, Inequality (5.5.6) reduces to

$$\begin{aligned} P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\right) &\leq \delta n \frac{C'}{m} \\ &\leq \delta C_2 \quad \text{for some constant } C_2 > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{1 \leq \rho_n \leq \delta n} P\bigg(\frac{|\Delta_{n,m_n}|}{n} > \epsilon \mid m_n - m = \rho_n\bigg) P(m_n - m = \rho_n) \\ \leq \;& \delta C_2 \sum_{1 \leq \rho_n \leq \delta n} P(m_n - m = \rho_n) \\ = \;& \delta C_2 P(1 \leq m_n - m \leq \delta n) \\ \leq \;& \delta C_2. \end{aligned}$$

Substituting the above into Inequality (5.5.5), we get

$$P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \leq P\left(|m_n - m| > \delta n\right) + (C_1 + C_2)\delta. \tag{5.5.7}$$

Since $\frac{m}{n} \to r$, when n is sufficiently large, we have

$$\frac{|m - rn|}{n} \leq \delta/2,$$

thus

$$\begin{aligned} P\left(|m_n - m| > \delta n\right) &= P\left(\frac{|m_n - m|}{n} > \delta\right) \\ &\leq P\left(\frac{|m_n - rn|}{n} + \frac{|m - rn|}{n} > \delta\right) \\ &\leq P\left(\frac{|m_n - rn|}{n} > \frac{\delta}{2}\right) = P\left(|m_n - rn| \geq \delta n/2\right). \end{aligned}$$

Applying a similar argument by DeVinney and Wierman [12, page 432], which uses Chernoff's theorem, the above is further bounded as follows:

$$\begin{aligned} P\left(|m_n - m| > \delta n\right) &\leq P\left(|m_n - rn| \geq \frac{\delta n}{2}\right) \\ &\leq Ke^{-k\delta n} \quad \text{for some constants } K, k > 0. \end{aligned}$$

Once the above is substituted into Inequality (5.5.7), it follows that

$$P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \leq Ke^{-k\delta n} + (C_1 + C_2)\delta.$$

For any fixed $\delta > 0$, the first term $Ke^{-k\delta n}$ goes to 0 as $n \to \infty$. Also, considering $\delta > 0$ can be arbitrarily small, we conclude that $P\left(\frac{|\Delta_{n,m_n}|}{n} > \epsilon\right) \to 0$, thus $\frac{\Delta_{n,m_n}}{n} \to 0$ in probability. □

In Theorem 5.5.1, the exact form of $g(r)$ is not given; however, it does have the following properties.

Corollary 5.5.1. *$g(r)$ is a bounded, increasing and continuous function on $(0, \infty)$.*

Proof.

- First, we show $g(r) \in [0, 1]$. For integer n, we have showed that

  $$\lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} = g(r) \quad a.s.,$$

  where $\lambda_X = 1$ and $\lambda_Y = r$ are assumed.

  Since $0 \leq \frac{\bar{\Gamma}_{J_n}}{|J_n|} \leq \frac{N_X(J_n)}{|J_n|}$ and $\frac{N_X(J_n)}{|J_n|} \xrightarrow{a.s} \lambda_X = 1$, it follows that

  $$0 \leq g(r) = \lim_{n\to\infty} \frac{\bar{\Gamma}_{J_n}}{|J_n|} \leq 1.$$

- Next, we show that $g(r)$ increases as r increases. We first suppose that, for any $0 < r_1 < r_2$, there is a Poisson process X with rate 1, a Poisson process Y_1 with rate r_1, and another Poisson process Y_{2-1} with rate $r_2 - r_1$. Then for any integer $n > 0$, we let $t(n)$ be the smallest real number t such that there are $n+1$ X-points in J_t. Suppose next that $m_1(n)$ is the random number of Y_1-points in $J_{t(n)}$, and $m_{2-1}(n)$ is the random number of Y_{2-1}-points in $J_{t(n)}$. We refer to both the Y_1-points and Y_{2-1}-points as Y_2-points. We define $\Gamma_{n,m_1(n)}$ as the domination number generated by the X-points and Y_1-points in $J_{t(n)}$, and $\Gamma_{n,m_2(n)}$ as the domination number generated by the X-points and Y_2-points in $J_{t(n)}$. Basically, we have just added $m_{2-1}(n)$ Y_{2-1}-points to those m_1 Y_1-points to allow us to study the change from $\Gamma_{n,m_1(n)}$ to $\Gamma_{n,m_2(n)}$. Considering that adding Y-points can never decrease the domination number, we know that $\Gamma_{n,m_2(n)}$ is larger than $\Gamma_{n,m_1(n)}$. Note that Y_1-points are generated from a Poisson process with rate r_1, and Y_{2-1}-points are generated from a Poisson process with rate $r_2 - r_1$, hence Y_2-points are generated from a Poisson process with rate r_2. Therefore, by previous results, we have

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|} = g(r_1) \quad \text{a.s.,}$$

and

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|} = g(r_2) \quad \text{a.s.}$$

Recalling $\Gamma_{n,m_2(n)}$ is larger than $\Gamma_{n,m_1(n)}$, we conclude that $g(r_2) \geq g(r_1)$.

- Finally, all that remains is to prove that $g(r)$ is continuous. In other words, for any $r_1 > 0$ and $\epsilon > 0$, we must show that there exists a $\delta \equiv \delta(\epsilon) > 0$ such that when $|r_2 - r_1| < \delta$,

$$|g(r_2) - g(r_1)| \leq \epsilon.$$

Suppose there is a Poisson process X with rate 1, a Poisson process Y_1 with rate r_1, and another Poisson process Y_2 with rate r_2. Then for any integer $n > 0$, we let $t(n)$ be the smallest real number t such that there are $n+1$ X-points in $J_{t(n)}$. Suppose next that $m_1(n)$ is the random number of Y_1-points in $J_{t(n)}$, and $m_2(n)$ is the random number of Y_2-points in $J_{t(n)}$. Taking into consideration that almost sure convergence implies convergence in probability, we have

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|} = g(r_1) \quad \text{in probability,}$$

and

$$\lim_{n\to\infty} \frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|} = g(r_2) \quad \text{in probability.}$$

Next, we will prove $|g(r_2) - g(r_1)| \leq \epsilon$ by contradiction. Suppose $|g(r_2) - g(r_1)| = \epsilon + \alpha$ for $\alpha > 0$. By the definition of convergence in probability, we know that when n is sufficiently large,

$$P\left(\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|} - g(r_1)\right| < \alpha/2\right) > 1 - \delta,$$

and

$$P\left(\left|\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|} - g(r_2)\right| < \alpha/2\right) > 1 - \delta.$$

On intersection of the 2 events above, we have

$$\begin{aligned}\epsilon+\alpha &= |g(r_2)-g(r_1)| \\ &\leq \left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-g(r_1)\right|+\left|\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}-g(r_2)\right|+\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}\right| \\ &< \alpha/2+\alpha/2+\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}\right|,\end{aligned}$$

which can be reduced to $\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}\right|>\epsilon$. Since this intersection event has probability greater than $1-2\delta$, we conclude that

$$P\left(\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}\right|>\epsilon\right)>1-2\delta. \tag{5.5.8}$$

On the other hand, applying the same techniques used in the proof of Lemma 5.5.1, we can prove the following result similar to Inequality (5.5.7). If $|r_2-r_1|<\delta$, then when n is sufficiently large,

$$\begin{aligned}&P\left(\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}\right|>\epsilon\right) \\ \leq\ & P\left(|m_2(n)-m_1(n)|\geq\delta n\right)+(C_1'+C_2')\delta,\end{aligned}$$

where $C_1',C_2'>0$ are two constants determined by ϵ. Applying Chernoff's theorem as before, we know that for any fixed $\delta>0$, $P\left(|m_2(n)-m_1(n)|\geq\delta n\right)$ converges to 0 as n goes to $+\infty$. Therefore, when δ is sufficiently small, the inequality above can be further bounded as follows:

$$\begin{aligned}P\left(\left|\frac{\Gamma_{n,m_1(n)}}{|J_{t(n)}|}-\frac{\Gamma_{n,m_2(n)}}{|J_{t(n)}|}\right|>\epsilon\right) &\leq C\delta \qquad \text{(for some constant } C>0) \\ &\leq 1-2\delta,\end{aligned}$$

which contradicts Inequality (5.5.8). □

5.6 WLLN for the Domination Number in $[0,1]^2$ with General Densities

Based on the same approach used to prove the SLLN for the domination number with continuous densities in one dimension (Theorem 2.3.1 in Chapter 2), the following is also true.

Theorem 5.6.1. *If the densities f_X and f_Y are positive, bounded and continuous on $[0,1]^2$, and $\frac{m}{n} \to r, r \in (0,\infty)$, then*

$$\lim_{n\to\infty} \frac{\Gamma'_{n,m}}{n} = \iint_{[0,1]^2} g\left(r \cdot \frac{f_Y(u,v)}{f_X(u,v)}\right) \cdot f_X(u,v)\,du\,dv \quad \textit{in probability,}$$

where $g(r)$ is the same as in Theorem 5.5.1.

In the following two subsections, we first generalize Theorem 5.5.1 to the the case for piecewise constant densities, then extend it to the continuous case. The proofs are analogous to Sections 2.2 and 2.3. The only significant difference is that adding or deleting a point in two dimensions, rather than changing the domination number by at most 2, could change the domination number quite substantially (as much as $n-1$). Nonetheless, such large changes are highly unlikely and are proved to be negligible in the limit.

5.6.1 Piecewise Constant Densities

We consider the simple situation in which f_X and f_Y are piecewise constant densities defined as

$$f_X(u,v) = \sum_{p,q=1}^{k} a_{pq} I_{A_{pq}}(u,v)$$

and

$$f_Y(u,v) = \sum_{p,q=1}^{k} b_{pq} I_{A_{pq}}(u,v),$$

where $A_{pq}, p, q = 1, \cdots, k$ equally divide $[0,1]^2$ into k^2 pieces (see Figure 5.7). Let

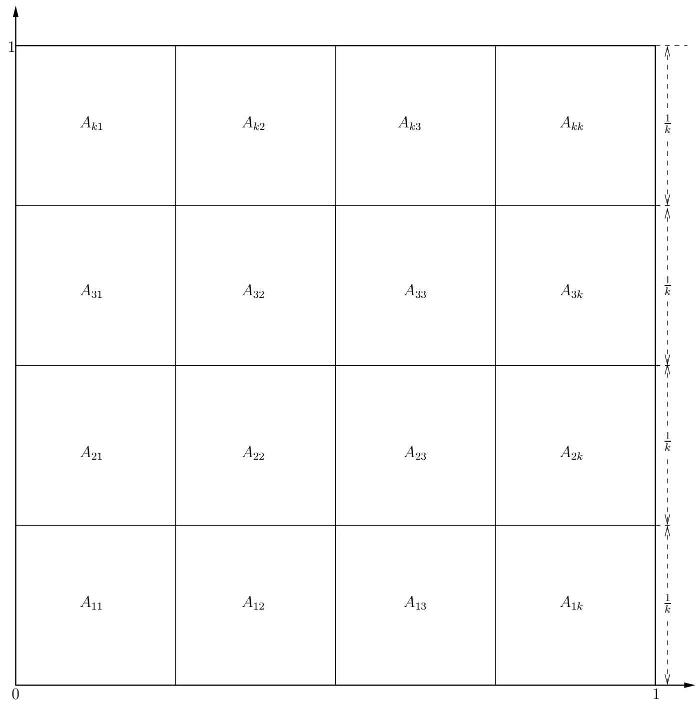

Figure 5.7: The squares A_{pq} where densities f_X and f_Y are constant

$\Gamma(n,m)$ be the domination number generated by the n X-points and m Y-points in

$[0,1]^2$, and $\Gamma(n_{pq}, m_{pq})$ be the domination number generated by the n_{pq} X-points and m_{pq} Y-points in A_{pq}. We can think of $\sum_{p,q}\Gamma(n_{pq}, m_{pq})$ as a "filtered" domination number generated by adding a "filter" A_{pq} for each $\Gamma(n_{pq}, m_{pq})$. The effect of adding "a filter" is that no points outside A_{pq} contribute to $\Gamma(n_{pq}, m_{pq})$. The outcome of ignoring the filters is the restoration of the sum of the "filtered" domination numbers $\sum_{p,q}\Gamma(n_{pq}, m_{pq})$ to the ordinary domination number $\Gamma(n,m)$. We define

$$\Delta_{n,m} = \Gamma_{n,m} - \sum_{p,q}\Gamma(n_{pq}, m_{pq}).$$

We now apply the same technique used in Section 5.3. Specifically, with $d = d(n)$ to be chosen later, we shrink each A_{pq} by $\delta = 1/kd$ to get A'_{pq}, then shrink A'_{pq} by δ to get A''_{pq}, and then shrink A''_{pq} by $\sqrt{2}\delta$ to get A'''_{pq} (see Figure 5.8). Finally, we divide $A'_{pq} - A''_{pq}$ equally into $4d-12$ small squares with side length δ. Now that there are $(4d-12)$ small squares in each $A_{pq}, p, q = 1, \cdots, k$, there are totally $(4d-12)k^2$ small squares in $\cup A_{p,q}$. Define F_m={there exists at least one Y-point in each small square}. Then we have

$$\begin{aligned} P(F_m) &= \left(1 - \left(1-\delta^2\right)^m\right)^{(4d-12)k^2} \\ &\geq \left(1 - \left(1-(1/kd)^2\right)^m\right)^{4dk^2}. \end{aligned} \tag{5.6.1}$$

Next, we will apply the results obtained in the proof of the SLLN of the ordinary domination number in the Poisson case (refer to Figures 5.2 and 5.3). Conditional upon F_m, the covering ball of any X-point in A''_{pq} is contained in A_{pq}. Therefore, ignoring the filter A_{pq} has no effect on these X-points. However, there may be some

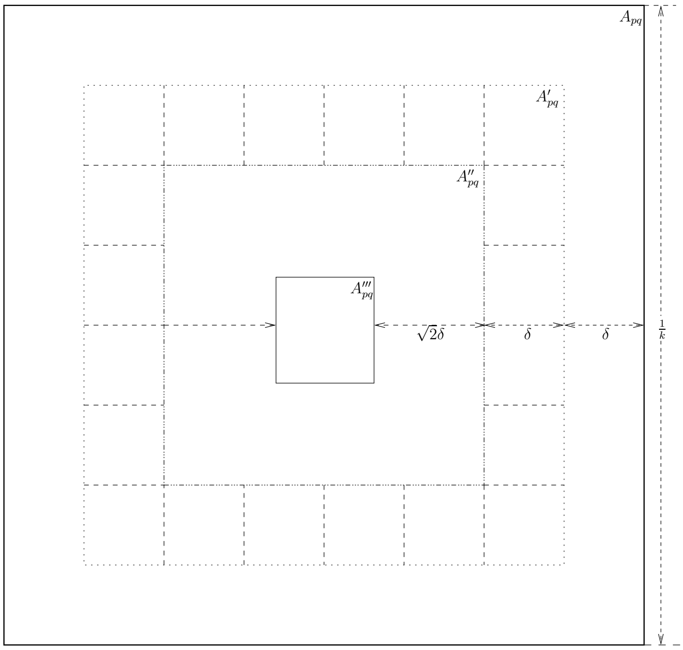

Figure 5.8: The rectangles $A_{pq}, A'_{pq}, A''_{pq}$ and A'''_{pq}

Y-points just outside the boundary of A_{pq}, while some X-point in $A_{pq} - A''_{pq}$ could have a covering ball that is not contained in A_{pq}. Thus, ignoring the filter A_{pq} could reduce the covering ball of some X-points in $A_{pq} - A''_{pq}$, thereby increasing the domination number. Such an increase is bounded by the number of X-points in $A_{pq} - A'''_{pq}$, since no covering ball of any X-point in $A_{pq} - A''_{pq}$ can intersect with A'''_{pq}. Summarizing the argument above, we get

$$\begin{aligned} \frac{\Gamma_{n,m}}{n} &= \sum_{p,q} \frac{\Gamma(n_{pq}, m_{pq})}{n} + \frac{\Delta_{n,m}}{n} \\ &= \sum_{p,q} \frac{\Gamma(n_{pq}, m_{pq})}{n} + \frac{\Delta_{n,m}}{n} I_{F_m^C} + \frac{\Delta_{n,m}}{n} I_{F_m}, \qquad (5.6.2) \end{aligned}$$

where

$$\left|\frac{\Delta_{n,m}}{n} I_{F_m^C}\right| \leq I_{F_m^C} \tag{5.6.3}$$

and

$$0 \leq \frac{\Delta_{n,m}}{n} I_{F_m} \leq \frac{\sum_{p,q} N_X(A_{pq} - A'''_{pq})}{n}. \tag{5.6.4}$$

In the rest of this section, we will calculate the limit of the three terms in the RHS of Equation (5.6.2). First, applying a similar argument to that used in the proof of Lemma 2.2.1, we show that as $n \to \infty$,

$$\begin{aligned} \frac{m_{pq}}{m} &\to b_{pq}|A_{pq}| \quad a.s., \\ \frac{n_{pq}}{n} &\to a_{pq}|A_{pq}| \quad a.s., \end{aligned}$$

hence $\frac{m_{pq}}{n_{pq}} \to r_{pq}$ $a.s.$, where $r_{pq} \equiv r \cdot \frac{b_{pq}}{a_{pq}}$. Therefore, applying Theorem 5.5.1 on each A_{pq} yields

$$\frac{\Gamma(n_{pq}, m_{pq})}{n} \to g(r_{pq}) \cdot a_{pq}|A_{pq}| \quad \text{in probability,}$$

and thus

$$\sum_{p,q} \frac{\Gamma(n_{pq}, m_{pq})}{n} \to \sum_{p,q} g(r_{pq}) \cdot a_{pq}|A_{pq}| \quad \text{in probability.}$$

Writing the above in the form of an integral gives

$$\begin{aligned} \sum_{p,q} \frac{\Gamma(n_{pq}, m_{pq})}{n} &\to \sum_{p,q} g\left(r \cdot \frac{b_{pq}}{a_{pq}}\right) \cdot a_{pq}|A_{pq}| \\ &= \iint_{[0,1]^2} g\left(r \cdot \frac{f_Y(u,v)}{f_X(u,v)}\right) \cdot f_X(u,v)\,du\,dv \quad \text{in probability.} \end{aligned} \tag{5.6.5}$$

Next, from Inequality (5.6.1), we know that when d is sufficiently large,

$$\begin{aligned} 0 \geq logP(F_m) &\geq 4dk^2 \cdot log\Big(1 - \big(1 - (\frac{1}{kd})^2\big)^m\Big) \\ &= -4k^2 d \cdot \bigg(\big(1 - \frac{1}{k^2d^2}\big)^m + o\Big(\big(1 - \frac{1}{k^2d^2}\big)^m\Big)\bigg). \end{aligned}$$

Since $(1 - \frac{1}{x})^x \nearrow e^{-1}$ as $x \to \infty$, it follows that

$$\begin{aligned} -4k^2 d \cdot \left(1 - \frac{1}{k^2d^2}\right)^m &= -4k^2 d \cdot \left(1 - \frac{1}{k^2d^2}\right)^{k^2d^2 \cdot \frac{m}{k^2d^2}} \\ &\geq -4k^2 d \cdot e^{-\frac{m}{k^2d^2}}. \end{aligned}$$

Let $d = \left\lfloor \sqrt{\frac{m}{k^2 \cdot log\sqrt{m}}} \right\rfloor$. Then

$$\begin{aligned} -4k^2 d \big(1 - \frac{1}{k^2d^2}\big)^m &\geq -4k^2 \sqrt{\frac{m}{k^2 \cdot log\sqrt{m}}} \cdot \frac{1}{\sqrt{m}} \\ &= -\frac{4k}{\sqrt{log\sqrt{m}}}, \end{aligned}$$

and thus

$$\begin{aligned} 0 \geq logP(F_m) &\geq -\frac{4k}{\sqrt{log\sqrt{m}}} + o\left(-\frac{4k}{\sqrt{log\sqrt{m}}}\right) \\ &\overset{m\to\infty}{\longrightarrow} 0. \end{aligned}$$

It also follows that $P(F_m) \to 1$, so $P(F_m^C) \to 0$. Hence,

$$I_{F_m^C} \to 0 \quad \text{in probability.}$$

The formula above combined with Inequality (5.6.3) gives

$$\frac{\Delta(n,m)}{n} I_{F_{n,m}^C} \to 0 \quad \text{in probability.} \tag{5.6.6}$$

Finally, we write the RHS in Inequality (5.6.4) as

$$\frac{\sum_{p,q} N_X(A_{pq} - A'''_{pq})}{n} = \frac{\sum_{i=1}^{n} I\big(X_i \in \cup_{p,q}(A_{pq} - A'''_{pq})\big)}{n}.$$

It should be noted that

$$\begin{aligned}
& E\Big[I\big(X_i \in \cup_{p,q}(A_{pq} - A'''_{pq})\big)\Big] \\
= \;& P\big(X_i \in \cup_{p,q}(A_{pq} - A'''_{pq})\big) \\
= \;& 1 - (1 - 2(2+\sqrt{2})\delta)^{2k^2} \quad \text{(refer to Figure 5.7 and Figure 5.8).}
\end{aligned}$$

Recall that $\delta = 1/kd$ and $d = \left\lfloor \sqrt{\frac{m}{k^2 \cdot log\sqrt{m}}} \right\rfloor$. It follows that when m is sufficiently large,

$$\begin{aligned}
E\Big[I\big(X_i \in \cup_{p,q}(A_{pq} - A'''_{pq})\big)\Big] \quad & \leq \quad 1 - \left(1 - 2(2+\sqrt{2}) \cdot \frac{1}{\sqrt{\frac{m}{log\sqrt{m}}} - 1}\right)^{2k^2} \\
& \stackrel{m\to\infty}{\longrightarrow} \quad 0.
\end{aligned}$$

Therefore, for any $\delta > 0$, the Markov inequality provides

$$\begin{aligned}
P\left(\frac{\sum_{p=1}^{n} I\big(X_i \in \cup_{p,q}(A_{pq} - A'''_{pq})\big)}{n} \geq \delta\right) \quad & \leq \quad \frac{E\Big[I\big(X_i \in \cup_{p,q}(A_{pq} - A'''_{pq})\big)\Big]}{\delta} \\
& \to \quad 0,
\end{aligned}$$

and thus

$$\frac{\sum_{p,q} N_X(A_{pq} - A'''_{pq})}{n} \to 0 \quad \text{in probability.}$$

Combining the above with Inequality (5.6.4) yields

$$\frac{\Delta(n,m)}{n} I_{F_m} \to 0 \quad \text{in probability.} \tag{5.6.7}$$

Finally, substituting Formulas (5.6.5), (5.6.6) and (5.6.7) into (5.6.2) generates the desired result.

Remark 5.6.1. The proof above can be easily generalized to the case when the regions of constancy for the densities are rectangles instead of squares. However, the limiting function g then depends on the ratio between the length and the width of the rectangles, hence the final limiting value can't be written in the simple integral form.

5.6.2 Continuous Densities

If f_X and f_Y are bounded and continuous, then they are both uniformly continuous on $[0,1]^2$. Thus, given any $\delta > 0$, there exists an integer k_0 such that for any $k \geq k_0$ and the equal partition $\{A_{pq}, p, q = 1, \cdots, k\}$ of $[0,1]^2$ (refer to Figure 5.7), the following must hold:

$$|f_X(u_1, v_1) - f_X(u_2, v_2)| \leq \delta,$$

$$|f_Y(u_1, v_1) - f_Y(u_2, v_2)| \leq \delta,$$

for any $(u_1, v_1), (u_2, v_2) \in A_{pq}$. We define piecewise constant functions approximating f_X and f_Y as follows:

$$\bar{f}_X(u, v) = \min\{f_X(x, y), (x, y) \in A_{pq}\} \quad \text{for } (u, v) \in A_{pq},$$

$$\bar{f}_Y(u, v) = \min\{f_Y(x, y), (x, y) \in A_{pq}\} \quad \text{for } (u, v) \in A_{pq},$$

and then rescale $\bar{f}_X$ and $\bar{f}_Y$ by dividing them by their respective integrals to give piecewise constant densities $\hat{f}_X$ and $\hat{f}_Y$, which approximate f_X and f_Y, respectively.

Applying the same argument as in Section 2.3 to i.i.d. random vectors $(X_{i1}, X_{i2}, X_{i3}), i = 1, \cdots, n$, distributed uniformly between $\{(u, v, 0) : u, v \in [0,1]^2\}$ and the surface $\big\{\big(u, v, f_X(u, v)\big) : u, v \in [0,1]^2\big\}$, the marginal distribution of (X_{i1}, X_{i2}) is proved to be f_X. The same procedure generates i.i.d. random vectors (Y_{j1}, Y_{j2}, Y_{j3}), $j = 1, \cdots, m$, with the marginal distribution of (Y_{j1}, Y_{j2}) being f_Y.

Next, we let $(\bar{X}_{i1}, \bar{X}_{i2}, \bar{X}_{i3}), i = 1, \cdots, n$, and $(\bar{Y}_{j1}, \bar{Y}_{j2}, \bar{Y}_{j3}), j = 1, \cdots, m$, be i.i.d. random vectors uniformly distributed under surface $\{(u, v, \bar{f}_X(u, v)) : u, v \in [0,1]^2\}$ and $\big\{\big(u, v, \bar{f}_Y(u, v)\big) : u, v \in [0,1]^2\big\}$, respectively.

Finally, we define $\bar{R}_X$ as the region between the surfaces $\big\{\big(u, v, f_X(u, v)\big) : u, v \in [0,1]^2\big\}$ and $\big\{\big(u, v, \bar{f}_X(u, v)\big) : u, v \in [0,1]^2\big\}$, and $\bar{R}_Y$ as the region between the surfaces $\big\{\big(u, v, f_Y(u, v)\big) : u, v \in [0,1]^2\big\}$ and $\big\{\big(u, v, \bar{f}_Y(u, v)\big) : u, v \in [0,1]^2\big\}$. We then define

$$
\begin{aligned}
(\hat{X}_{i1}, \hat{X}_{i2}, \hat{X}_{i3}) &= (X_{i1}, X_{i2}, X_{i3})\; I_{\{(X_{i1}, X_{i2}, X_{i3}) \notin \bar{R}_X\}} \\
&\quad + (\bar{X}_{i1}, \bar{X}_{i2}, \bar{X}_{i3})\; I_{\{(X_{i1}, X_{i2}, X_{i3}) \in \bar{R}_X\}}, \\
(\hat{Y}_{j1}, \hat{Y}_{j2}, \hat{Y}_{j3}) &= (Y_{j1}, Y_{j2}, Y_{j3})\; I_{\{(Y_{j1}, Y_{j2}, Y_{j3}) \notin \bar{R}_Y\}} \\
&\quad + (\bar{Y}_{j1}, \bar{Y}_{j2}, \bar{Y}_{j3})\; I_{\{(Y_{j1}, Y_{j2}, Y_{j3}) \in \bar{R}_Y\}}.
\end{aligned}
$$

The sequences above can be reiterated as follows. For each $i \in \{1, \cdots, n\}$, if the point (X_{i1}, X_{i2}, X_{i3}) falls into $\bar{R}_X$, then this point is defined as $(\hat{X}_{i1}, \hat{X}_{i2}, \hat{X}_{i3})$; otherwise,

$(\bar{X}_{i1}, \bar{X}_{i2}, \bar{X}_{i3})$ is defined as $(\hat{X}_{i1}, \hat{X}_{i2}, \hat{X}_{i3})$. A similar procedure applies to Y-points. Again, using a similar technique to that in Section 2.3 shows that $(\hat{X}_{i1}, \hat{X}_{i2})$ and $(\hat{Y}_{i1}, \hat{Y}_{i2})$ have piecewise constant densities $\hat{f}_X$ and $\hat{f}_Y$, respectively.

Define $\mathcal{X} \equiv \{\mathcal{X}_i = (X_{i1}, X_{i2}), i = 1, \cdots, n\}$, and $\mathcal{Y} \equiv \{\mathcal{Y}_j = (Y_{j1}, Y_{j2}), j = 1, \cdots, m\}$. Let $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ denote the domination number generated by $\mathcal{X}$ and $\mathcal{Y}$. Similarly, let $\Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})$ denote the domination number generated by $\hat{\mathcal{X}} \equiv \{\hat{\mathcal{X}}_i = (\hat{X}_{i1}, \hat{X}_{i2}), i = 1, \cdots, n\}$, and $\hat{\mathcal{Y}} \equiv \{\hat{\mathcal{Y}}_j = (\hat{Y}_{j1}, \hat{Y}_{j2}), j = 1, \cdots, m\}$. Note that only the points $(X_{i1}, X_{i2}, X_{i3}) \in \bar{R}_X$ and $(Y_{j1}, Y_{j2}, Y_{j3}) \in \bar{R}_Y$ could cause the difference between $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ and $\Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})$. Such difference could be as much as $n - 1$. However, by applying the results obtained in Section 5.5.1, we will next show that if the largest covering ball is small, then the difference is negligible in the limit.

When any $X_i = (X_{i1}, X_{i2}, X_{i3}) \in \bar{R}_X$ is replaced by $\bar{X}_i = (\bar{X}_{i1}, \bar{X}_{i2}, \bar{X}_{i3})$, it is equivalent to deleting $\mathcal{X}_i = (X_{i1}, X_{i2})$ and then adding $\bar{\mathcal{X}}_i = (\mathcal{X}_{i1}, \mathcal{X}_{i2})$. Deleting $\mathcal{X}_i$ could decrease (but never increase) the domination number $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ by at most 1. On the other hand, note that deleting the covering ball of $\mathcal{X}_i$ could also increase (but never decrease) the domination number by at most the number of $\mathcal{X}$-points in $B(\mathcal{X}_i) - \{\mathcal{X}_i\}$. Hence, deleting $\mathcal{X}_i$ could change the domination number by at most the number of $\mathcal{X}$-points in $B(\mathcal{X}_i)$. Similarly, adding $\bar{\mathcal{X}}_i$ could further increase the domination number by at most 1. However, note that adding the covering ball of $\bar{\mathcal{X}}_i$ can also decrease the domination number by at most the number of $\hat{\mathcal{X}}$-points in $B(\bar{\mathcal{X}}_i) - \{\bar{\mathcal{X}}_i\}$. Hence, adding $\bar{\mathcal{X}}_i$ can change the domination number by at most the number of $\hat{\mathcal{X}}$-points in $B(\bar{\mathcal{X}}_i)$. Therefore, replacing any $X_i \in \bar{R}_X$ by $\bar{X}_i$ could only change the domination number by at most

$$\sum_{l=1}^{n} I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + \sum_{l=1}^{n} I_{\{\bar{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}}.$$

Hence, the change caused by X_i in R_X is bounded by

$$\sum_{i=1}^{n} I_{\{X_i \in \bar{R}_X\}} \sum_{l=1}^{n} \left(I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + \sum_{l=1}^{n} I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}} \right). \tag{5.6.8}$$

Given any $X_i \in \bar{R}_X$, we denote the radius of the covering ball $B(\mathcal{X}_i)$ by B_i. For any fixed $X_i \in \bar{R}_X$, we can bound $P(B_i > b)$ as follows:

$$\begin{aligned} & P(B_i > b \mid X_i, X_i \in \bar{R}_X) \\ \leq \;& P(\text{there are no } \mathcal{Y}\text{-points in the ball centered at } \mathcal{X}_i \text{ with radius } b). \end{aligned}$$

Recall that f_X, $\hat{f}_X$, f_Y and $\hat{f}_Y$ are all positive and bounded, so we can assume $k_1 \leq f_X \leq k_2$, $k_1 \leq \hat{f}_X \leq k_2$, $k_1 \leq f_Y \leq k_2$ and $k_1 \leq \hat{f}_Y \leq k_2$ for some positive constants k_1, k_2. Hence, the inequality above can be further bounded as

$$P(B_i > b \mid X_i, X_i \in \bar{R}_X) \;\leq\; \left(1 - k_1 \pi b^2\right)^m.$$

Since the bound above is uniform for any $X_i \in \bar{R}_X$, it follows that

$$P(B_i > b \mid X_i \in \bar{R}_X) \leq \left(1 - k_1 \pi b^2\right)^m.$$

Note that for any $l \in \{1, \cdots, i-1, i+1, \cdots, n\}$, the random point $\mathcal{X}_l$ is independent of X_i and $Y_j, j = 1, \cdots, m$. Therefore,

$$\begin{aligned}
E\left[I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} \mid X_i \in \bar{R}_X\right] &= E\Big[E\left[I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} \mid X_i \in \bar{R}_X, B_i\right] \mid X_i \in \bar{R}_X\Big] \\
&= E\Big[P\left(\mathcal{X}_l \in B(\mathcal{X}_i) \mid X_i \in \bar{R}_X, B_i\right) \mid X_i \in \bar{R}_X\Big] \\
&\leq E\left[k_2 \pi B_i^2 \mid X_i \in \bar{R}_X\right].
\end{aligned}$$

By applying the same technique as on page 88, we can further bound the above as follows:

$$\begin{aligned}
E\left[I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} \mid X_i \in \bar{R}_X\right] &\leq \int \left(1 - k_1 \pi b^2\right)^m d\left(k_2 \pi b^2\right) \\
&\leq \frac{1}{m+1}.
\end{aligned}$$

Since $m/n \to r$, when n is sufficiently large, it follows that

$$\begin{aligned}
E\left[\sum_{l=1}^{n} I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} \mid X_i \in \bar{R}_X\right] &= 1 + E\left[\sum_{l \in \{1, \cdots, i-1, i+1, \cdots, n\}} I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} \mid X_i \in \bar{R}_X\right] \\
&\leq 1 + \frac{n-1}{m+1} \\
&\leq K_1 \quad \text{for some constant } K_1.
\end{aligned}$$

Similarly, we can prove that when n is sufficiently large,

$$E\left[\sum_{l=1}^{n} I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}} \mid X_i \in \bar{R}_X\right] \leq K_1.$$

From the two inequalities above, we can bound the expectation of Formula (5.6.8) as follows:

$$
\begin{aligned}
& E\left[\sum_{i=1}^{n} I_{\{X_i \in \bar{R}_X\}} \sum_{l=1}^{n} \Big(I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}} \Big)\right] \\
= \;& \sum_{i=1}^{n} E\left[I_{\{X_i \in \bar{R}_X\}} \sum_{l=1}^{n} \Big(I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}} \Big)\right] \\
= \;& \sum_{i=1}^{n} E\left[\sum_{l=1}^{n} \Big(I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}} \Big) \mid X_i \in \bar{R}_X \right] P(X_i \in \bar{R}_X) \\
\leq \;& \sum_{i=1}^{n} 2K_1 P(X_i \in \bar{R}_X) \\
= \;& 2K_1 \sum_{i=1}^{n} P(X_i \in \bar{R}_X) \\
\leq \;& 2K_1 \delta n. \qquad (5.6.9)
\end{aligned}
$$

After replacing all $X_i \in \bar{R}_X$ by $\bar{X}_i$, the original domination number $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ becomes $\Gamma_{n,m}(\hat{\mathcal{X}}, \mathcal{Y})$. Next, we consider the effect of replacing $Y_j = (Y_{j1}, Y_{j2}, Y_{j3}) \in \bar{R}_Y$ by $\bar{Y}_j = (\bar{Y}_{j1}, \bar{Y}_{j2}, \bar{Y}_{j3})$, which is equivalent to deleting $\mathcal{Y}_j = (Y_{j1}, Y_{j2})$ and then adding $\bar{\mathcal{Y}}_j = (\bar{Y}_{j1}, \bar{Y}_{j2})$. We have discussed the effect of deleting and adding $\mathcal{Y}$-points in Section 5.5.1. For all $Y_j \notin \bar{R}_Y$, refer to the corresponding points $\mathcal{Y}_j$ as $\mathcal{Y}'$-points. For any $Y_j \in \bar{R}_Y$, define $B_j \equiv$ maximum radius of all balls that contain $\mathcal{Y}_j$ but contain no $\mathcal{Y}'$-points. Applying the arguments in Section 5.5.1 shows that deleting $\mathcal{Y}_j$ could increase (but never decrease) $\Gamma_{n,m}(\hat{\mathcal{X}}, \mathcal{Y})$ by at most the number of $\hat{X}$-points in the ball $B(\mathcal{Y}_j) \equiv B(\mathcal{Y}_j, 2B_j)$, centered at $\mathcal{Y}_j$ with radius $2B_j$. Furthermore, for any $Y_j \in \bar{R}_Y$, define $\bar{B}_j \equiv$ maximum radius of all balls that contain $\bar{\mathcal{Y}}_j$ but contain no $\mathcal{Y}'$-points. Similarly, applying the arguments in Section 5.5.1 shows that adding $\bar{\mathcal{Y}}_j$ could further decrease (but never increase) $\Gamma_{n,m}(\hat{\mathcal{X}}, \mathcal{Y})$ by at most the number of

$\hat{X}$-points in $B(\bar{\mathcal{Y}}_j) \equiv B(\bar{\mathcal{Y}}_j, 2\bar{B}_j)$, centered at $\bar{\mathcal{Y}}_j$ with radius $2\bar{B}_j$. Thus, replacing any $Y_j \in \bar{R}_Y$ by $\bar{Y}_j$ could further change the original domination number $\Gamma_{n,m}(\mathcal{X}, \mathcal{Y})$ by no more than

$$\sum_{j=1}^{m} I_{\{Y_j \in \bar{R}_Y\}} \sum_{l=1}^{n} \left(I_{\{\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{Y}}_j)\}} \right). \tag{5.6.10}$$

Let m_R denote the number of $Y_j \in \bar{R}_Y$. For any fixed $Y_j \in \bar{R}_Y$, using the same argument as on page 90, we can bound $P(B_j > b \mid Y_j, Y_j \in \bar{R}_Y, m_R)$ as follows:

$$\begin{aligned} & P(B_j > b \mid Y_j, Y_j \in \bar{R}_Y, m_R) \\ \le\ & 64 \cdot P(\text{there are no } \mathcal{Y}'\text{-points in a particular grid ball}) \\ \le\ & 64 \left(1 - k_2 \pi (b/4)^2\right)^{m - m_R}. \end{aligned}$$

Since the bound above is uniform for any $Y_j \in \bar{R}_Y$, it follows that

$$P(B_j > b \mid Y_j \in \bar{R}_Y, m_R) \ \le\ 64 \left(1 - k_2 \pi (b/4)^2\right)^{m - m_R}.$$

Note that for any $l \in \{1, \cdots, n\}$, the random point $\hat{\mathcal{X}}_j$ is independent of Y_j, $j = 1, \cdots, m$. Therefore,

$$\begin{aligned} & E\left[I_{\{\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j)\}} \mid Y_j \in \bar{R}_Y, m_R \right] \\ =\ & E\left[E\left[I_{\{\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j)\}} \mid Y_j \in \bar{R}_Y, m_R, B_j \right] \mid Y_j \in \bar{R}_Y, m_R \right] \\ =\ & E\left[P\left(\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j) \mid Y_j \in \bar{R}_Y, m_R, B_j \right) \mid Y_j \in \bar{R}_Y, m_R \right] \\ \le\ & E\left[k_2 \pi (2B_j)^2 \mid Y_j \in \bar{R}_Y, m_R \right]. \end{aligned}$$

By applying the same technique as on page 88, we can further bound the above as follows

$$E\left[I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}} \mid Y_j\in \bar{R}_Y, m_R\right] \le \int 64\left(1-k_1\pi(b/4)^2\right)^{m-m_R} d\big(k_2\pi(2b)^2\big).$$

Hence,

$$\begin{aligned}E\left[\sum_{l=1}^{n} I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}} \mid Y_j\in \bar{R}_Y, m_R\right] &\le n\cdot \int 64\left(1-k_1\pi(b/4)^2\right)^{m-m_R} d\big(k_2\pi(2b)^2\big)\\ &\le n\cdot\frac{C}{m-m_R} \quad \text{for some constant } C>0.\end{aligned}$$

Since $m/n \to r$, when n is sufficiently large, conditional on $m_R \le 2\delta m$ the inequality above yields

$$\begin{aligned}E\left[\sum_{l=1}^{n} I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}} \mid Y_j\in \bar{R}_Y, m_R\le 2\delta m\right] &\le n\cdot\frac{C}{m-2\delta m}\\ &\le K_2 \;\; \text{for some constant } K_2>0. \quad (5.6.11)\end{aligned}$$

Furthermore, by applying the argument above to the case of adding $\bar{\mathcal{Y}}_j$, we conclude that, when n is sufficiently large,

$$E\left[\sum_{l=1}^{n} I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}} \mid Y_j\in \bar{R}_Y, m_R\le 2\delta m\right] \le K_2. \quad (5.6.12)$$

Note that

$$\begin{aligned}&E\left[\sum_{j=1}^{m} I_{\{Y_j\in\bar{R}_Y\}}\sum_{l=1}^{n}\Big(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\Big) \mid m_R\le 2\delta m\right]\\ =\;& \sum_{j=1}^{m} E\left[I_{\{Y_j\in\bar{R}_Y\}}\sum_{l=1}^{n}\Big(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\Big) \mid m_R\le 2\delta m\right]\\ =\;& \sum_{j=1}^{m} E\left[\sum_{l=1}^{n}\Big(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\Big) \mid Y_j\in\bar{R}_Y, m_R\le 2\delta m\right]\\ &\cdot P\left(Y_j\in\bar{R}_Y \mid m_R\le 2\delta m\right)\end{aligned}$$

Applying Inequalities (5.6.11) and (5.6.12) to the above equation, we obtain

$$\begin{aligned}
& E\left[\sum_{j=1}^{m} I_{\{Y_j \in \bar{R}_Y\}} \sum_{l=1}^{n} \left(I_{\{\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{Y}}_j)\}} \right) \mid m_R \leq 2\delta m \right] \\
\leq \; & 2K_2 \sum_{j=1}^{m} P\left(Y_j \in \bar{R}_Y \mid m_R \leq 2\delta m\right) \\
= \; & 2K_2 \sum_{j=1}^{m} E\left[I_{\{Y_j \in \bar{R}_Y\}} \mid m_R \leq 2\delta m\right] \\
= \; & 2K_2 E\left[\sum_{j=1}^{m} I_{\{Y_j \in \bar{R}_Y\}} \mid m_R \leq 2\delta m\right] \\
= \; & 2K_2 E\left[m_R \mid m_R \leq 2\delta m\right] \leq 4K_2 \delta m. \qquad (5.6.13)
\end{aligned}$$

Recall from Formulas (5.6.8) and (5.6.10) that $\left|\Gamma_{n,m}(\mathcal{X}, \mathcal{Y}) - \Gamma_{n,m}(\hat{\mathcal{X}}, \hat{\mathcal{Y}})\right|$ is bounded by

$$\begin{aligned}
& \sum_{i=1}^{n} I_{\{X_i \in \bar{R}_X\}} \sum_{l=1}^{n} \left(I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{X}}_i)\}} \right) \\
+ \; & \sum_{j=1}^{m} I_{\{Y_j \in \bar{R}_Y\}} \sum_{l=1}^{n} \left(I_{\{\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{Y}}_j)\}} \right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& P\left(\left|\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})-\Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n}\right|>\epsilon\right) \\
= \; & P\left(\left|\Gamma_{n,m}(\mathcal{X},\mathcal{Y})-\Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})\right|>\epsilon n\right) \\
\leq \; & P\left(\sum_{i=1}^{n} I_{\{X_i\in\bar{R}_X\}}\sum_{l=1}^{n}\left(I_{\{\mathcal{X}_l\in B(\mathcal{X}_i)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{X}}_i)\}}\right)>n\epsilon/2\right) \\
& + P\left(\sum_{j=1}^{m} I_{\{Y_j\in\bar{R}_Y\}}\sum_{l=1}^{n}\left(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\right)>n\epsilon/2\right) \\
= \; & P\left(\sum_{i=1}^{n} I_{\{X_i\in\bar{R}_X\}}\sum_{l=1}^{n}\left(I_{\{\mathcal{X}_l\in B(\mathcal{X}_i)\}}+I_{\{(\hat{\mathcal{X}}_l)\in B(\bar{\mathcal{X}}_i)\}}\right)>n\epsilon/2\right) \\
& +P\left(\sum_{j=1}^{m} I_{\{Y_j\in\bar{R}_Y\}}\sum_{l=1}^{n}\left(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\right)>n\epsilon/2 \mid m_R\leq 2\delta m\right) P(m_R\leq 2\delta m) \\
& +P\left(\sum_{j=1}^{m} I_{\{Y_j\in\bar{R}_Y\}}\sum_{l=1}^{n}\left(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\right)>n\epsilon/2 \mid m_R>2\delta m\right) P(m_R> 2\delta m) \\
= \; & P\left(\sum_{i=1}^{n} I_{\{X_i\in\bar{R}_X\}}\sum_{l=1}^{n}\left(I_{\{\mathcal{X}_l\in B(\mathcal{X}_i)\}}+I_{\{(\hat{\mathcal{X}}_l)\in B(\bar{\mathcal{X}}_i)\}}\right)>n\epsilon/2\right) \\
& +P\left(\sum_{j=1}^{m} I_{\{Y_j\in\bar{R}_Y\}}\sum_{l=1}^{n}\left(I_{\{\hat{\mathcal{X}}_l\in B(\mathcal{Y}_j)\}}+I_{\{\hat{\mathcal{X}}_l\in B(\bar{\mathcal{Y}}_j)\}}\right)>n\epsilon/2 \mid m_R\leq 2\delta m\right)+P(m_R>2\delta m).
\end{aligned}
$$

With Formulas (5.6.9) and (5.6.13), applying the Markov Inequality yields that, for any $\epsilon > 0$, when n is sufficiently large,

$$
\begin{aligned}
& P\left(\left|\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y}) - \Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n}\right| > \epsilon\right) \\
\leq \; & \frac{E\left[\sum\limits_{i=1}^{n} I_{\{X_i \in \bar{R}_X\}}\left(\sum\limits_{l=1}^{n} I_{\{\mathcal{X}_l \in B(\mathcal{X}_i)\}} + I_{\{\hat{\mathcal{X}}_l) \in B(\bar{\mathcal{X}}_i)\}}\right)\right]}{n\epsilon/2} \\
& + \frac{E\left[\sum\limits_{j=1}^{m} I_{\{Y_j \in \bar{R}_Y\}}\left(\sum\limits_{l=1}^{n} I_{\{\hat{\mathcal{X}}_l \in B(\mathcal{Y}_j)\}} + I_{\{\hat{\mathcal{X}}_l \in B(\bar{\mathcal{Y}}_j)\}}\right) \mid m_R \leq 2\delta m\right]}{n\epsilon/2} + P(m_R > 2\delta m) \\
\leq \; & \frac{2K_1\delta n}{n\epsilon/2} + \frac{4K_2\delta m}{n\epsilon/2} + P(m_R > 2\delta m) \\
\leq \; & K\delta + P(m_R > 2\delta m) \quad \text{for some constant } K \text{ determined by } \epsilon. \qquad (5.6.14)
\end{aligned}
$$

Note that m_R is a binomial random variable with mean δm and variance $\delta(1-\delta)m$, so by applying the Markov Inequality we get

$$
\begin{aligned}
P(m_R > 2\delta m) &= P(m_R - \delta m > \delta m) \\
&\leq \frac{\delta(1-\delta)m}{(\delta m)^2} = \frac{(1-\delta)}{\delta m}.
\end{aligned}
$$

Thus, for any fixed $\delta \in (0,1)$, when m is sufficiently large, the following inequality holds:

$$P(m_R > 2\delta m) \leq \delta.$$

Hence, Inequality (5.6.14) reduces to

$$P\left(\left|\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y}) - \Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n}\right| > \epsilon\right) \leq K\delta + \delta. \qquad (5.6.15)$$

In the previous section, we have proved that

$$\frac{\Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n} \to \iint\limits_{[0,1]^2} g\left(r \cdot \frac{\hat{f}_Y(u,v)}{\hat{f}_X(u,v)}\right) \cdot \hat{f}_X(u,v)\,du\,dv \quad \text{in probability.}$$

Thus, when n is sufficiently large,

$$P\left(\left|\frac{\Gamma_{n,m}(\hat{\mathcal{X}},\hat{\mathcal{Y}})}{n} - \iint\limits_{[0,1]^2} g\left(r\cdot\frac{\hat{f}_Y(u,v)}{\hat{f}_X(u,v)}\right)\cdot\hat{f}_X(u,v)\,du\,dv\right| > \epsilon\right) \leq \delta.$$

Combining the inequality above with Inequality (5.6.15) gives

$$P\left(\left|\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})}{n} - \iint\limits_{[0,1]^2} g\left(r\cdot\frac{\hat{f}_Y(u,v)}{\hat{f}_X(u,v)}\right)\cdot\hat{f}_X(u,v)\,du\,dv\right| > \epsilon\right) \leq K\delta + 2\delta.$$

Corollary 5.5.1 says that $g(r)$ is bounded and continuous. Since $\hat{f}_X \to f_X$ and $\hat{f}_Y \to f_Y$ as $\delta \to 0$, then by the dominated convergence theorem it follows that

$$\iint\limits_{[0,1]^2} g\left(r\cdot\frac{\hat{f}_Y(u,v)}{\hat{f}_X(u,v)}\right)\cdot\hat{f}_X(u,v)\,du\,dv \to \iint\limits_{[0,1]^2} g\left(r\cdot\frac{f_Y(u,v)}{f_X(u,v)}\right)\cdot f_X(u,v)\,du\,dv.$$

Considering $\delta > 0$ can be arbitrarily small, we immediately obtain

$$P\left(\left|\frac{\Gamma_{n,m}(\mathcal{X},\mathcal{Y})}{n} - \iint\limits_{[0,1]^2} g\left(r\cdot\frac{f_Y(u,v)}{f_X(u,v)}\right)\cdot f_X(u,v)\,du\,dv]\right| > \epsilon\right) \to 0.$$

This finishes the proof of Theorem 5.6.1.

Remark 5.6.2. The limiting function $\iint\limits_{[0,1]^2} g\left(r\cdot\frac{f_Y(u,v)}{f_X(u,v)}\right)\cdot f_X(u,v)\,du\,dv$ gives the same value as for uniform densities whenever $f_X = f_Y$. But since we haven't proved whether g is concave, we don't know yet if this limiting function achieves the maximum value when $f_X = f_Y$.

Chapter 6

Monte Carlo Simulation

In previous chapters, we have proved the general SLLN and the uniform CLT for the domination number in the one-dimensional case, and established the WLLN in two dimensions. In this chapter, we will use Monte Carlo simulations to illustrate these theoretical results, and also empirically verify some limit theorems that are not obtained in this dissertation but are likely to be true, such as the CLT in two dimensions.

6.1 One Dimension

In the one-dimensional case, we use Monte Carlo simulations to generate n X-points and m Y-points according to distribution functions F_X and F_Y, respectively; then we calculate the domination number induced by these X-points and Y-points. For each of four combinations of F_X and F_Y, the simulations are repeated 1000 times.

(All of the above is programmed in Matlab.) The results for these four combinations are described in the following four subsections, respectively.

6.1.1 The Case of $F_X = F_Y = U[0, 1]$

By using R, the generated box-plots of $\frac{\Gamma_{n,m}}{n}$ are shown in Figure 6.1. The graph indicates that the sample mean of $\frac{\Gamma_{n,m}}{n}$ converges to a constant as $m = n \to \infty$, which is consistent with Theorem 2.3.1. Specifically, from Table 6.1, we can see that when $m = n = 10000$, the sample mean is very close to the theoretical limiting value of $\frac{\Gamma_{n,m}}{n}$, which is about 0.59524 according to Theorem 2.3.1. Furthermore, we notice that the sample mean decreases as n increases. Such phenomenon also exists for other combinations of F_X and F_Y, as illustrated by the simulation results in next three subsections. We don't have an explanation as to why the sample mean monotonically decreases to the theoretical limit.

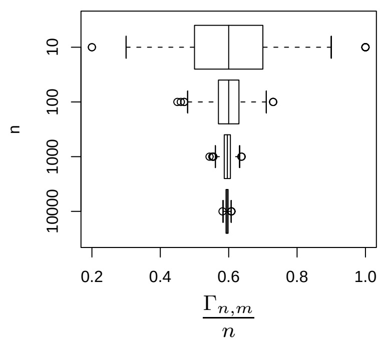

Figure 6.1: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000, 10000$ from top to bottom, when $F_X = F_Y = U[0, 1]$.

$n(=m)$	10	100	1000	10000
sample mean of $\Gamma_{n,m}/n$	0.63280	0.59716	0.59583	0.59527

Table 6.1: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X = F_Y = U[0,1]$.

Also, in Figure 6.2, we compare the sample variance of the domination number with the theoretical limiting variance of the domination number given in Theorem 3.4.2. For every $r \in \{0.01, 0.02, \cdots, 0.3, 0.4, \cdots, 2\}$, we calculate the sample variance $s(r)$ for a sample of 10000 domination numbers, each generated by n X-points and m Y-points uniformly distributed in $[0,1]$ $\big($with $\min\{n,m\} = 1000$ and $m = \lfloor rn \rfloor\big)$. In Figure 6.2, each dot has coordinates $(r, s(r)/m)$, and the continuous curve is the graph of $v(r)$ given in Theorem 3.4.2. We see that the dots fit well with the graph of $v(r)$, which confirms our result obtained in Chapter 3.

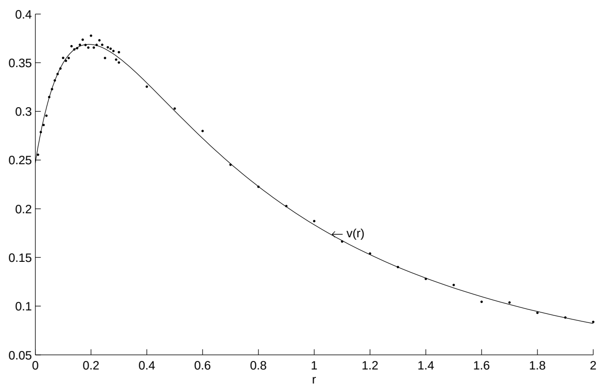

Figure 6.2: The sample variance of the domination number compared with the theoretical limiting variance of the domination number, when $F_X = F_Y = U[0,1]$.

Next, we plot the histograms and normal qq-plots of $\Gamma_{n,m}$ using R, as shown in Figure 6.3. Both histograms and normal qq-plots support the CLT for the domination number generated by one-dimensional uniform data, which has been proven in Chapter 4.

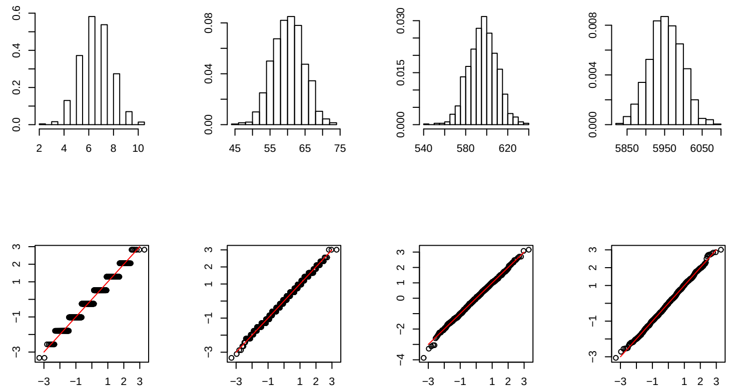

Figure 6.3: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100$, 1000, 10000 from left to right, when $F_X = F_Y = U[0, 1]$.

Finally, to quantify the rate of convergence to normality, we run the Shapiro-Wilk test [22] on $\Gamma_{n,m}$ for several typical values of m, n. Because the Shapiro-Wilk test runs on an empirical sample and tests the normality of the sample's underlying distribution, the p-values only estimate the convergence rate. Table 6.2 summarizes our experimental outputs. The rate of convergence to normality is fast in that the p-value increases from < 0.01 to > 0.10 when $m = n$ increases from 100 to 300.

$n(=m)$	10	100	200	300	1000	10000
p-value	$< 2.2e - 16$	0.001446	0.01304	0.1396	0.3131	0.6807

Table 6.2: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X = F_Y = U[0, 1]$.

6.1.2 The Case of $F_X = x^2, F_Y = y^2$ on $[0,1]$

When $F_X = x^2$ and $F_Y = y^2$ on $[0,1]$, from Figure 6.4 we see that the sample mean of $\frac{\Gamma_{n,m}}{n}$ still appears to converge to a constant. Furthermore, Table 6.3 suggests that the sample mean converges to the theoretical limiting value 0.59524, which is the same as in the case of $F_X = F_Y = U[0,1]$ because of Corollary 2.4.1. As in the previous case, the histograms and normal qq-plots (Figure 6.5) suggest that $\Gamma_{n,m}$ is still approximately normal. The p-values listed in Table 6.4 imply the rate of convergence to normality might be as fast as the case of $F_X = F_Y = U[0,1]$.

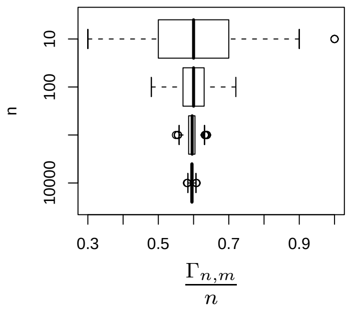

Figure 6.4: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100$, 1000, 10000 from top to bottom, when $F_X = x^2$ and $F_Y = y^2$ on $[0,1]$.

$n(=m)$	10	100	1000	10000
sample mean of $\Gamma_{n,m}/n$	0.63180	0.59854	0.59566	0.59536

Table 6.3: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X = x^2$ and $F_Y = y^2$ on $[0,1]$.

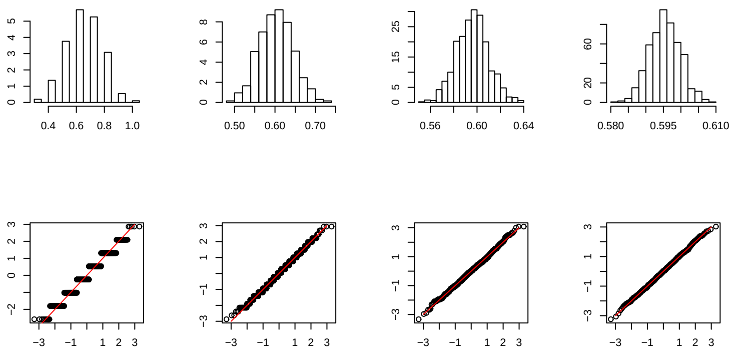

Figure 6.5: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100$, 1000, 10000 from left to right, when $F_X = x^2$ and $F_Y = y^2$ on $[0, 1]$.

$n(= m)$	10	100	200	300	1000	10000
p-value	$< 2.2e - 16$	0.0002023	0.02491	0.04427	0.2732	0.905

Table 6.4: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X = x^2$ and $F_Y = y^2$ on $[0, 1]$.

6.1.3 The Case of $F_X = U[0, 1], F_Y = y^2$ on $[0, 1]$

In this case, we consider two nonequal densities. The SLLN (Theorem 2.3.1) should still apply, and this is supported by the box-plots shown in Figure 6.6. The sample mean listed in Table 6.5 appears to converge quickly to the theoretical limiting value 0.53346, which is less than the theoretical limiting value 0.59524 for equal densities according to Corollary 2.4.1. The histograms and normal qq-plots (Figure 6.7) imply that the domination number is still asymptotically normal.

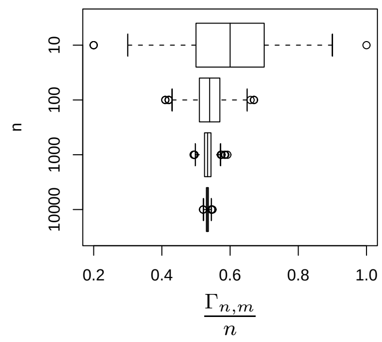

Figure 6.6: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000, 10000$ from top to bottom, when $F_X = U[0, 1]$ and $F_Y = y^2$ on $[0, 1]$.

$n(= m)$	10	100	1000	10000
sample mean of $\Gamma_{n,m}/n$	0.5825	0.53941	0.53427	0.5334844

Table 6.5: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X = U[0, 1]$ and $F_Y = y^2$ on $[0, 1]$.

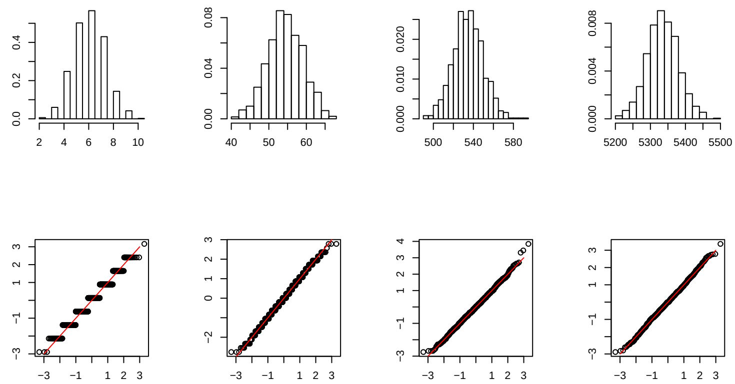

Figure 6.7: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100$, 1000, 10000 from left to right, when $F_X = U[0,1], F_Y = y^2$ on $[0,1]$.

$n(=m)$	10	100	200	300	1000	10000
p-value	$< 2.2e-16$	0.001162	0.01502	0.1020	0.2017	0.6633

Table 6.6: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X = U[0,1]$ and $F_Y = y^2$ on $[0,1]$.

6.1.4 The Case of $F_X = .25 * x * I_{\{x \in [0,.5)\}} + (1.5 * x - .5) * I_{\{x \in [.5,1]\}}$ and $F_Y = y^2$ on $[0,1]$

Finally we check the case of piecewise constant densities (the linear piecewise distribution function above corresponds to a piecewise constant density function). Again, the box-plots (Figure 6.8) validate the SLLN, and Table 6.7 suggests the theoretical limiting value is 0.57567, less than the theoretical limiting value 0.59524 for equal densities according to Corollary 2.4.1. The CLT is empirically checked in Figure 6.9 and Table 6.8.

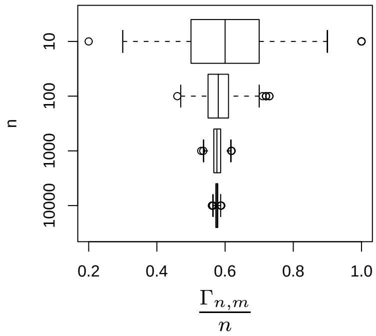

Figure 6.8: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000, 10000$ from top to bottom, when $F_X = .25 * x * I_{\{x\in[0,.5)\}} + (1.5 * x - .5) * I_{\{x\in[.5,1]\}}, F_Y = y^2$ on $[0, 1]$.

$n(= m)$	10	100	1000	10000
sample mean of $\Gamma_{n,m}/n$	0.62180	0.58353	0.576326	0.575754

Table 6.7: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X = .25*x*I_{\{x\in[0,.5)\}}+(1.5*x-.5)*I_{\{x\in[.5,1]\}}$ and $F_Y = y^2$ on $[0, 1]$.

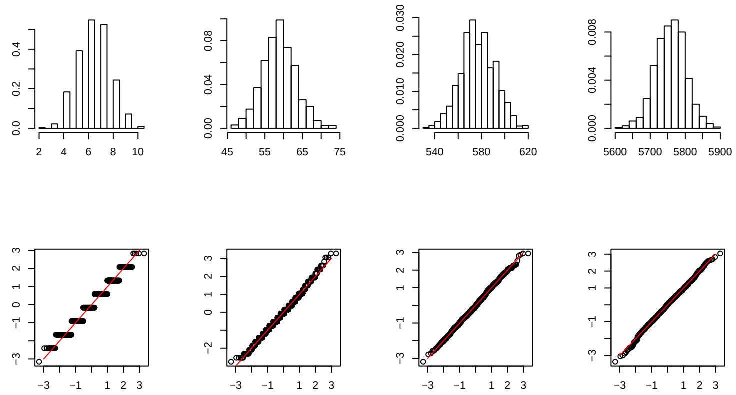

Figure 6.9: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100$, 1000, 10000 from left to right, when $F_X = .25 * x * I_{\{x\in[0,.5)\}} + (1.5 * x - .5) * I_{\{x\in[.5,1]\}}$ and $F_Y = y^2$ on $[0, 1]$.

$n(= m)$	10	100	200	300	1000	10000
p-value	$< 2.2e - 16$	$9.28e - 05$	0.058	0.05571	0.3417	0.3572

Table 6.8: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X = .25*x*I_{\{x\in[0,.5)\}}+(1.5*x-.5)*I_{\{x\in[.5,1]\}}$ and $F_Y = y^2$ on $[0, 1]$.

6.2 Higher Dimensions

Because finding the dominating set in higher dimensional CCCDs is NP-hard, calculating the domination number is much more time consuming for higher dimensions than for one dimension. Thus, we have only simulated up to the case when $m = n = 1000$. In addition, we convert the problem of finding the domination number into an integer linear programming problem, as shown by Priebe et al. [3, page 245]. To solve these integer linear programming problems, we use the Matlab interface provided by *lp_solve*, a free linear programming solver. (For more information on the program, visit the *lp_solve* Web site [23].)

As in the one-dimensional case, we plot the box-plots, sample mean, histograms, normal qq-plots and p-values for each of the four combinations of densities, shown in the following four subsections, respectively.

As in the one-dimensional case, we plot the box-plots for different combinations of m, n, F_X and F_Y, shown in Figure 6.10, Figure 6.12, Figure 6.14 and Figure 6.16. The box-plots indicate that the sample mean of $\frac{\Gamma_{n,m}}{n}$ converges to a constant as $m = n \to \infty$, which supports Theorem 5.6.1. In particular, the sample means of $\frac{\Gamma_{m,n}}{n}$ are 0.57927 and 0.58000 for $F_X = F_Y = U[0,1]^2$ and $F_X(u,v) = F_Y(u,v) = u^2v^2$, respectively. This result again supports the formula in Theorem 5.6.1, which gives the same limiting value for any equal f_X and f_Y. Again, as in the one-dimensional case, we notice that the sample mean decreases as n increases. But we can't provide an explanation as to why the sample mean monotonically decreases to the theoretical limit.

The CLT in higher dimensions is not established in this dissertation. However, we use Monte Carlo simulations to check whether the CLT holds empirically. The generated histograms and normal qq-plots are plotted using R, shown in Figure 6.11, Figure 6.13, Figure 6.15 and Figure 6.17. The histograms and normal qq-plots show that $\Gamma_{n,m}$ is likely to be asymptotically normal.

Furthermore, we use p-values of the Shapiro-Wilk test to estimate the rate of convergence to normality. The outputs, summarized in Table 6.10, Table 6.12, Table 6.14 and Table 6.16, illustrate that when $m, n \geq 300$, the p-values are ≥ 0.1 for both uniform and nonuniform densities. This finding implies that $\Gamma_{n,m}$ is already approximately normal when $m, n \geq 300$. A comparison between the p-values reveals that for the same value of m and n, the convergence rate is slower in two dimensions than in one dimension. One possible reason is that because X-points and Y-points are more flexibly distributed in two dimensions, the domination number has a larger variance, which may cause a slower rate of convergence to normality.

6.2.1 The Case of $F_X = F_Y = U[0,1]^2$

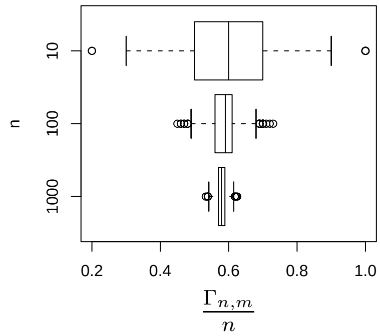

Figure 6.10: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000$ from top to bottom, when $F_X = F_Y = U[0,1]^2$.

$n(= m)$	10	100	1000
sample mean of $\Gamma_{n,m}/n$	0.61530	0.58578	0.57927

Table 6.9: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X = F_Y = U[0,1]^2$.

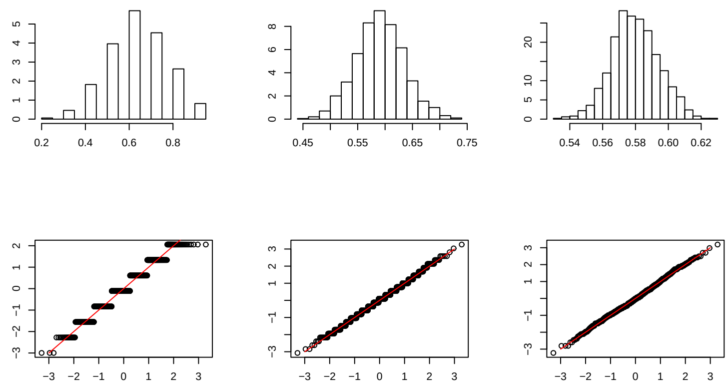

Figure 6.11: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100, 1000$ from left to right, when $F_X = F_Y = U[0, 1]^2$.

$n(= m)$	10	100	200	300	1000
p-value	$< 2.2e - 16$	0.002011	0.02087	0.08009	0.1561

Table 6.10: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X = F_Y = U[0, 1]^2$.

6.2.2 The Case of $F_X(u,v) = F_Y(u,v) = u^2v^2$ on $[0,1]^2$

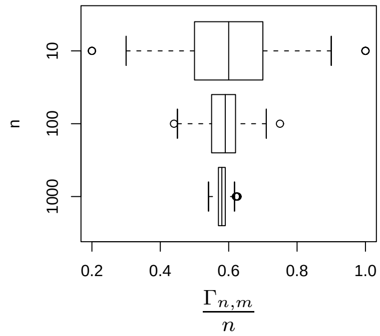

Figure 6.12: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000$ from top to bottom, when $F_X(u,v) = F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

$n(=m)$	10	100	1000
sample mean of $\Gamma_{n,m}/n$	0.61720	0.58560	0.58000

Table 6.11: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X(u,v) = F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

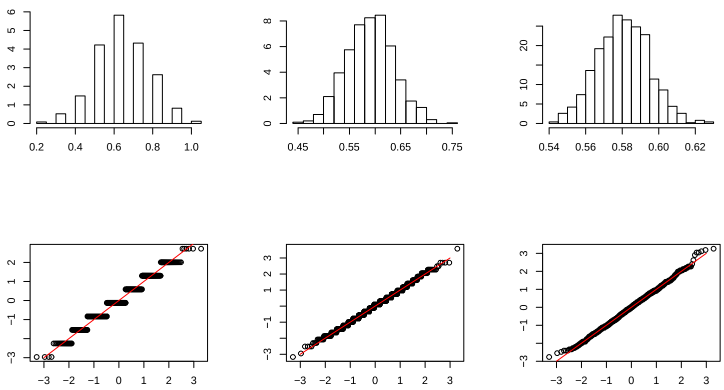

Figure 6.13: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100, 1000$ from left to right, when $F_X(u, v) = F_Y(u, v) = u^2v^2$ on $[0, 1]^2$.

$n(= m)$	10	100	200	300	1000
p-value	$< 2.2e - 16$	0.00227	0.02758	0.04500	0.11700

Table 6.12: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X(u, v) = F_Y(u, v) = u^2v^2$ on $[0, 1]^2$.

6.2.3 The Case of $F_X = U[0,1]^2, F_Y(u,v) = u^2v^2$ on $[0,1]^2$

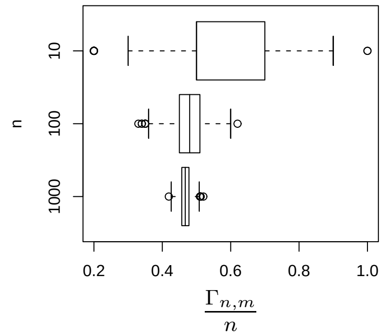

Figure 6.14: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000$ from top to bottom, when $F_X = U[0,1]^2, F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

$n(= m)$	10	100	1000
sample mean of $\Gamma_{n,m}/n$	0.55370	0.47900	0.46762

Table 6.13: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X = U[0,1]^2$ and $F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

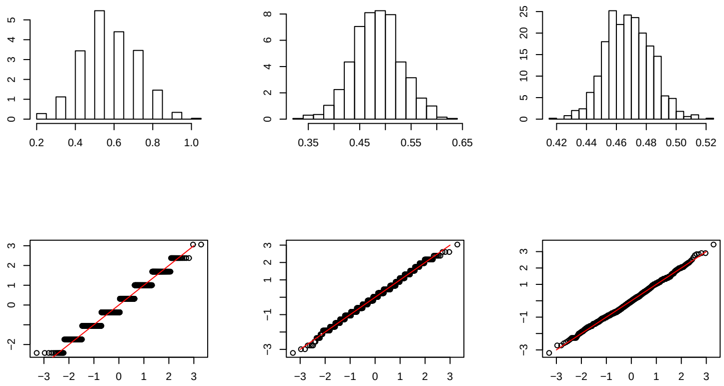

Figure 6.15: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100, 1000$ from left to right, when $F_X = U[0,1]^2$ and $F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

$n(=m)$	10	100	200	300	1000
p-value	$< 2.2e-16$	0.001248	0.01666	0.06904	0.08149

Table 6.14: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X = U[0,1]^2$ and $F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

6.2.4 The Case of $F_X(u,v) = \big(.25 * u * I_{\{u\in[0,.5)\}} + (1.5 * u - .5) * I_{\{u\in[.5,1]\}}\big) * \big(.25 * v * I_{\{v\in[0,.5)\}} + (1.5 * v - .5) * I_{\{v\in[.5,1]\}}\big)$ and $F_Y(u,v) = u^2v^2$ on $[0,1]^2$

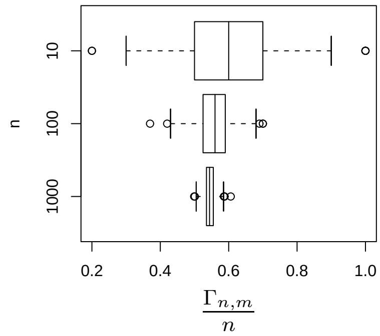

Figure 6.16: The box-plots of $\frac{\Gamma_{n,m}}{n}$ with $m = n = 10, 100, 1000$ from top to bottom, when $F_X(u,v) = \big(.25 * u * I_{\{u\in[0,.5)\}} + (1.5 * u - .5) * I_{\{u\in[.5,1]\}}\big) * \big(.25 * v * I_{\{v\in[0,.5)\}} + (1.5 * v - .5) * I_{\{v\in[.5,1]\}}\big)$ and $F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

$n(=m)$	10	100	1000
sample mean of $\Gamma_{n,m}/n$	0.62260	0.55796	0.54456

Table 6.15: The sample mean of $\frac{\Gamma_{n,m}}{n}$ when $F_X(u,v) = \big(.25 * u * I_{\{u\in[0,.5)\}} + (1.5 * u - .5) * I_{\{u\in[.5,1]\}}\big) * \big(.25 * v * I_{\{v\in[0,.5)\}} + (1.5 * v - .5) * I_{\{v\in[.5,1]\}}\big)$ and $F_Y(u,v) = u^2v^2$ on $[0,1]^2$.

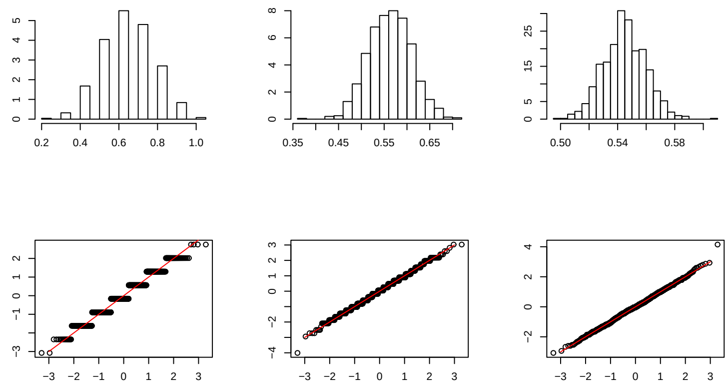

Figure 6.17: The histograms and normal qq-plots of $\Gamma_{n,m}$ with $m = n = 10, 100, 1000$ from left to right, when $F_X(u,v) = \big(.25 * u * I_{\{u \in [0,.5)\}} + (1.5 * u - .5) * I_{\{u \in [.5,1]\}}\big) * \big(.25 * v * I_{\{v \in [0,.5)\}} + (1.5 * v - .5) * I_{\{v \in [.5,1]\}}\big)$ and $F_Y(u,v) = u^2 v^2$ on $[0,1]^2$.

$n(= m)$	10	100	200	300	1000
p-value	$< 2.2e - 16$	0.002367	0.06357	0.2062	0.3634

Table 6.16: The p-values of the Shapiro-Wilk test for the null hypothesis that the domination number $\Gamma_{n,m}$ is normally distributed when $F_X(u,v) = \big(.25 * u * I_{\{u \in [0,.5)\}} + (1.5 * u - .5) * I_{\{u \in [.5,1]\}}\big) * \big(.25 * v * I_{\{v \in [0,.5)\}} + (1.5 * v - .5) * I_{\{v \in [.5,1]\}}\big)$ and $F_Y(u,v) = u^2 v^2$ on $[0,1]^2$.

Chapter 7

Discussion

In this dissertation, we have proved the SLLN for the domination number in one dimension with continuous and bounded densities. In addition, we have established the CLT for the domination number in one dimension with uniform densities. Finally, we have shown the SLLN for the domination number in the two-dimensional Poisson case, and the WLLN for the domination number in two dimensions with positive, bounded and continuous densities.

Our work in the one-dimensional case has set the stage for future research in higher dimensions. One direction would be to extend the CLT for the uniform density to continuous densities in one dimension, where we have encountered problems in calculating the variance of the domination number, and finally prove the CLT in higher dimensions. One possible approach to such an extension would be to apply the CLTs for certain graphs proven by Penrose and Yukich [24].

Another research direction would be to apply our methods to other properties of CCCDs, such as the edge density. For example, in the context of Chapter 5 of this dissertation, the edge number is likely to satisfy the subadditivity condition. Therefore, the proof of laws of large numbers for the domination number might be carried over to the edge number.

In summary, in this research, we have used previous results of other researchers and various tools such as negative association and subadditive processes to establish limit theory for the domination number of CCCDs in both one dimension and two dimensions. The techniques used in the development of the theorems also serve as a foundation for future improvements that could be directly applied to build and analyze CCCD classifiers.

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Vita

Pengfei Xiang was born on January 21, 1979 in Jianli County, Hubei Province, P.R. China. He finished the Jianli High School course in two years, and then he was accepted into the Special Class for the Gifted Young at University of Science and Technology of China (USTC) in September, 1994. He received his Bachelor's degree from USTC in July, 1999, majoring in Electronic Engineering and Information Science. During his last two years of college, Pengfei became interested in probability theory and statistics. This motivated him to attend the Johns Hopkins University in the fall of 1999 to pursue a Ph.D. in the Department of Applied Mathematics and Statistics. He obtained a Master's degree in Applied Mathematics and Statistics, as well as a Master's degree in Computer Science in May, 2001. Under the guidance of Professor John C. Wierman, he successfully finished his Ph.D. dissertation in December, 2005.

Since January of 2005, Pengfei has been employed as a Quantitative Associate by Majestic Research Corporation, an independent equity research firm.