Stochastic models for the coordinated production and shipment problem in a supply chain

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1. Introduction

In this study, we consider an integrated production and transportation problem between a single supplier and a single retailer in a stochastic environment. The supplier produces the items in a random amount of time, and sends them to the retailer to satisfy the external demand. Customers are willing to wait at the expense of a waiting cost. Consequently, the retailer does not hold inventory, instead he accumulates the orders to satisfy them at a later time when a shipment is received from the supplier. This kind of supply chains are typical for companies that carry on the sales process through sales agents, and for stores making catalog sales. Other examples include the supply chains involved with products that are unreasonable to keep in stock such as large items like photocopy machines, or luxury items like luxury watches or expensive sports cars. The retailers of such products are reluctant to keep them in stock since the inventory holding cost for the retailer is high, and the customer waiting cost is relatively low for a reasonable amount of waiting time. We assume that the supplier makes all the decisions, so that she decides when and how much to ship to the retailer, in addition to when and how much to produce.

Shipment consolidation and its economic consequences have been investigated in literature for more than 30 years. Our paper is closest to a subcategory of this literature, namely to the coordination of inventory and shipment decisions. For a review in this area, we refer to Çetinkaya (2004). Below, we review the studies most closely related to our paper.

We first consider the deterministic models. Çetinkaya and Lee (2002) consider a supply chain in which the supplier replenishes immediately, as opposed to producing the items, and the retailer does not hold inventory. They prove that the transportation cycle lengths in a replenishment cycle can be of two different time lengths. Kaya, Kubali, and Ormeci (2011) consider the same system in Çetinkaya and Lee (2002), except that the supplier produces the items instead of replenishing them instantly. In this case, the transportation cycles can be of three different lengths. Lee, Çetinkaya, and Jaruphongsa (2003) analyze optimal shipment policies with inventory lot sizing at a third-party warehouse. There are a number of studies which analyze the integrated production and shipment policies in a deterministic supply chain, where the retailer holds inventory and the supplier produces, see e.g., Hill (1999), Goyal (1995) and Lu (1995). Finally, the coordination of production and delivery schedules are considered in just-in-time systems in a deterministic environment, see e.g., Hahm and Yano (1995a, 1995b, 1995c).

When the demand arrivals are stochastic, finding and implementing optimal policies pose serious difficulties. Hence, existing literature identifies three types of simple consolidation policies: (1) time-based policies (2) quantity-based policies, and (3) time-and-quantity (TQ) based policies. In a time-based policy, orders are dispatched in every pre-determined time interval, \( T \), whereas under a quantity-based dispatch policy, a shipment is scheduled when the amount of the accumulated orders reaches a pre-determined consolidation quantity, \( Q \). In both of these policies,
shipments have to satisfy all the outstanding orders. TQ based policies are a hybrid of these two policies, so that they consolidate all orders until the earliest of a pre-determined shipping date, or a minimum pre-determined shipment quantity is reached.

Most of the initial work on the systems with random demand arrivals evaluated the performance of shipment consolidation practices, rather than optimizing the operational policy parameters, see e.g. Bookbinder and Higgins (2002), Çetinkaya and Lee (2000), Çetinkaya and Bookbinder (2003) derive analytical expressions of optimal policy parameters for time-based and quantity-based policies for the settings with private or common carriage opportunities. Mutlu, Bookbinder, and Çetinkaya (2010) extend the model of Çetinkaya and Lee (2000) to systems for which it is economical to use common carriage, rather than a private fleet of vehicles. Çetinkaya and Bookbinder (2003) derive analytical expressions of optimal policy parameters for time-based and quantity-based policies for the settings with private or common carriage opportunities. Mutlu, Bookbinder, and Çetinkaya (2010) extend the model of Çetinkaya and Lee (2000) to systems for which it is economical to use common carriage, rather than a private fleet of vehicles. Çetinkaya and Bookbinder (2003) derive analytical expressions of optimal policy parameters for time-based and quantity-based policies for the settings with private or common carriage opportunities. Mutlu, Bookbinder, and Çetinkaya (2010) extend the model of Çetinkaya and Lee (2000) to systems for which it is economical to use common carriage, rather than a private fleet of vehicles. Çetinkaya and Bookbinder (2003) derive analytical expressions of optimal policy parameters for time-based and quantity-based policies for the settings with private or common carriage opportunities. Mutlu, Bookbinder, and Çetinkaya (2010) extend the model of Çetinkaya and Lee (2000) to systems for which it is economical to use common carriage, rather than a private fleet of vehicles.

These studies analyzing stochastic models assume that the supplier can replenish immediately. In contrast, we consider a system where the items are produced by the supplier in a random amount of time. Hence, we consider the coordination of production (as opposed to inventory) and shipment decisions with the aim of minimizing the total holding and shipment costs. We analyze the simple consolidation policies, namely time-based, quantity-based and TQ-based policies that are analyzed in literature in the context of warehouse dispatching. In addition, we formulate and solve for an integrated optimal policy and compare the performances of all policies with the optimal one. The randomness in the production process results in a non-monotonic structure in the optimal policy. The production process in the system brings an additional dimension to the model and complicates the analysis. Even for the simple consolidation policies, the solutions cannot be obtained analytically, instead we optimize their parameters by numerical methods.

The information sharing in VMI settings can be carried on at different levels. We develop our first model for the case in which the retailer informs the supplier continuously about the number of outstanding orders. Given this information, the supplier decides on the timing and the quantity of the shipments, in addition to when to start and stop the production. We will refer to this model as the full-information model. Typically, it is not optimal to satisfy the orders immediately, but to let them accumulate for a while in order to satisfy the economies of scale, which results in significant cost savings in certain settings. Therefore, the first model of this study aims to answer the following questions: (i) When and how much should the supplier produce? (ii) When to dispatch a vehicle in order to satisfy the customer orders? (iii) In what quantity to dispatch so that economies of scale are satisfied? For this purpose, we formulate a Markov decision process (MDP) model, and observe that the resulting optimal policies behave in a non-monotonic way.

Next, we focus on the three heuristic policies analyzed in literature frequently, namely the time-based, quantity-based and TQ-based policies. These policies are also commonly-used in logistics industry, because of their easy implementation. The underlying information scheme under these policies corresponds to a partial information sharing, as the retailer shares only the probability distribution of the demand process with the supplier. We note that in this case, it is very difficult to determine the optimal solution under general conditions. Hence, we determine the optimal decision variables for the heuristic policies by constructing appropriate MDP models as a part of the optimization procedures. Finally, we numerically compare their performances with respect to the optimal solution in the full information case.

We start our analysis by presenting the full-information model for the system where the supplier has continuous access to the demand information of the retailer. Then, we present the models corresponding to the time-based, quantity-based and TQ-based dispatch policies in Sections 3–5, respectively. Section 6 numerically compares the performances of the three policies analytically. Finally, we conclude in Section 7.

2. Full-information model

We consider a single-supplier single-retailer supply chain system. The retailer does not hold inventory but accumulates orders to satisfy the customer’s demand, where customers are willing to wait at the expense of waiting cost. Waiting cost per unit time, denoted by $w$, is taken as a penalty associated with delayed shipment. The supplier produces and holds inventory. The supplier’s cost of carrying one unit of inventory per unit time is denoted by $h$. We assume that there is no relationship between $w$ and $h$, so our results are valid for all possible values of $h$ and $w$. We assume that an item is produced in an exponential amount of time with mean $1/\mu$, and the demand arrivals follow a Poisson process with an arrival rate of $\lambda$. These assumptions aim to model the randomness in demand and production, and are common in literature, see e.g., Carr and Duanyas (2000), de Vericourt, Karamen, and Dallery (2002), Örmeci, Burnetas, and Emmons (2002), and Gayon, Benjaafar, and de Vericourt (2009). The supplier incurs a setup cost, denoted as $K_s$, every time a production process starts. In addition, every time a shipment is dispatched to the retailer, a transportation cost, denoted as $K_t$, is incurred. We assume that the shipments are made with trucks of capacity $P$ and thus $\lceil Q/P \rceil$ trucks are required for each shipment, where $\lceil a \rceil$ is the ceiling of a real number $a$. The transportation cost includes a fixed cost $R$ that is independent of the shipment quantity, $Q$, and the cost of each truck, $S$, that depends on $Q$ such that $K_t = R + S/Q$. The transportation time between the supplier and the retailer is assumed to be negligible. Finally, we assume that the supplier has the option of outsourcing some of the products instead of producing them, and we let $s$ denote the additional per unit cost of outsourcing instead of producing them in-house. We note that this is without loss of generality for the full-information model, as $s$ can be chosen high enough if outsourcing is not allowed. However, as we will see later, it poses certain restrictions for the time-based, quantity-based and TQ-based models. Hence, we will provide an alternative interpretation for the cost of outsourcing in the context of these models.
The state of the system consists of three quantities: The inventory level of the supplier, \( I \), the total unsatisfied demand accumulated at the retailer, \( D \), and the state of the production, \( j \), where we set \( j = 1 \) when the production process is going on at the supplier, and \( j = 0 \) otherwise. Then, the system state is given by \((j,I,D)\), and the state space of the system is given by \( S \), where \( S = \{(j, I, D) : j \in \{0, 1\}, I \geq 0, D \geq 0\} \).

We can model this system as a continuous-time Markov decision process, with or without discounting and over a finite or an infinite horizon. Note that the problem without discounting over an infinite horizon has a solution only if \( \lambda < \mu \), which guarantees the existence of a stable system. In this model, we assume that \( \lambda < \mu \), and present a formulation with the objective of minimizing the expected long-run average cost. It can be shown that both \( I \) and \( D \) are bounded when an optimal policy is implemented. Hence, the state and action spaces as well as the cost are finite. These properties ensure that solutions to the long-run average problem always exist (see e.g., Puterman, 1994).

We use uniformization and normalization to present the discrete-time equivalent of this system, so we assume without loss of generality that \( \lambda = \mu = 1 \). Hence, the system will be observed at exponentially distributed intervals with mean 1. At each observation epoch a demand will arrive with probability \( \lambda \), and a production is completed with probability \( \mu \) if the production process is on \((j = 1)\); otherwise \((j = 0)\) a fictitious transition occurs with probability \( \mu \). Moreover, supplier must decide on dispatching and production at each of these epochs. More explicitly, when the system is in state \((j, I, D)\), the supplier decides on the dispatch quantity, \( Q \), and whether the production is set to be on \((j = 1)\) or off \((j = 0)\), which immediately changes the system state to \((j', l - Q^*), l - D^* \), where \((a') = \max(0,a)\) for any real number \( a \). We note that the dispatch quantity, \( Q \), cannot exceed the accumulated demand at the retailer, \( D \), since the retailer cannot hold inventory, so we always have \( 0 \leq Q \leq D \). There is no further restriction on the values of \( j \) and \( Q \).

We let \( g \) be the expected minimal long-run average cost of the system, and \( V(j, I, D) \) denote the relative value function evaluated when the production is in state \((j, I, D)\), the inventory level of the supplier is \( I \) and there are \( D \) customers waiting at the retailer. Furthermore, we define the auxiliary functions \( U(j, I, D, Q) \) in order to evaluate the cost of different action pairs \((j, Q)\) at each transition epoch. \( U(j, I, D, Q) \) computes the immediate cost of shipping \( Q \) units and being in state \((j', l - Q^*), l - D^* \) as well as the expected cost of future states determined by the next transitions.

We first consider the system when the production process is off, so \( j = 0 \). Then, the Bellman equations for state \((0, I, D)\) are given by:

\[
g + V(0, I, D) = \min_{0 \leq Q \leq D} \{U(1, I, D, Q) + K_p, U(0, I, D, Q))\}. \tag{2.1}
\]

If the production starts, so that \( j = 1 \), then the supplier must pay a set-up cost of \( K_p \), because the system started when \( j = 0 \). The functions \( U(j, I, D, Q) \) for \( j = 0 \), are as follows:

\[
U(j, I, D, Q) = s(Q - l^*) + h(I - Q^*) + w(D - Q) + \min_{0 \leq Q \leq D} \{1(Q > 0)K_p,
+ \lambda V(j, l - Q^*, D - Q + 1) + \mu V(j', l - Q'^* + j, D - Q), \tag{2.2}
\]

where \( I(A) \) is an indicator function, which is equal to 1 whenever the statement \( A \) is true, and equal to 0, otherwise. More explicitly:

\[
I(Q > 0) = \begin{cases} 1 & Q > 0, \\ 0 & Q = 0. \end{cases}
\]

If the supplier decides to dispatch more products than her current inventory level, \( I \), then she buys the extra amount from an ample supplier with a cost of \( s \) per unit leading to the cost term \( s(Q - l^*) \). The second and third terms of (2.2) incurs the holding and waiting cost in state \((j', l - Q'^*), D - Q)\), respectively. Whenever a shipment is dispatched, i.e., \( Q > 0 \), the transportation cost, \( K_p \), is incurred, leading to the term \( 1(Q > 0)K_p \).

In the case of a new demand arrival, which happens with probability \( \lambda \), the number of customers waiting at the retailer, \( D - Q \), increases by 1. The completion of the production, with probability \( \mu \), increases the inventory level at the supplier to \((l - Q'^*) + 1\) if the production process is on \((j = 1)\); otherwise \((j = 0)\) it corresponds to a fictitious state transition so that the inventory level remains as \((l - Q'^*)\).

For state \((1, I, D)\), the mathematical model is similar:

\[
V(1, I, D) = \min_{0 \leq Q \leq D} \{U(1, I, D, Q), U(0, I, D, Q))\}. \tag{2.3}
\]

In this case, the supplier may decide to continue the production or stop it, and if she decides to stop the production, then \( j \) changes from 1 to 0. The rest of the terms in Eq. (2.3) are similar to their counterparts in Eq. (2.1).

The following example shows the non-monotonic behavior of the optimal policy:

**Example 1.** In this example, we use the parameters \( \lambda = 0.3, \mu = 0.7, K_p = 600, K_t = 150, h = 2, w = 8 \) and \( s = 350 \). Fig. 1 presents the optimal actions \((j, Q)\) in all states when \( 0 < I < 20 \) and \( 0 < D < 20 \). More explicitly, Fig. 1a and b presents the dispatch sizes, \( Q \), for \( j = 0 \) and \( j = 1 \), respectively, while Fig. 1c and d show whether the production is turned off \((j = 0)\) or on \((j = 1)\), respectively.

We observe that it is never optimal to subcontract in this example, as the optimal policy ships at most the inventory level, \( I \). This is due to the relatively high value of subcontracting cost, \( s \). When both \( I \) and \( D \) are relatively high, the optimal policy always ships the minimum of \( I \) and \( D \), \( \min(I,D) \), in all states. However, when they are lower, the shipment policy depends on \( I \) and \( D \) in a non-monotone way.

We first consider the case when the production is off \((j = 0)\) and \( D = 3 \). The supplier chooses to wait for another demand arrival before shipment when the inventory level is 4, i.e., no shipment is scheduled in state \((0,4,3) \). However, she ships \( \min(I,D) \) in states with \( 2 < I < 3 \) and \( I \geq 5 \). Similarly, for \( j = 1 \) and \( D = 7 \), the supplier chooses to postpone a shipment until production of another item is completed when the inventory level is 6, i.e., in state \((1,6,7) \), while a shipment is dispatched in all states with \( 4 < I < 5 \) and \( I > 7 \).

Now we look at the optimal policy in a different way by focusing on \( j = 0 \) and \( I = 3 \). When \( D < 3 \), the optimal policy waits for more demand arrivals in order to dispatch all units in the inventory. For \( D = 3 \), all three items in the inventory are dispatched. However, if the demand level has reached a level of 5, the supplier turns on the production and waits until she produces at least one more item before the next shipment. Finally, we consider the system which starts with \( j = 1 \) and \( I = 4 \). The supplier dispatches the required demand when \( D = 3 \) or 4. If \( D \) is at a level of 5 or 6, then she delays the shipment with the hope of producing 1 or 2 more items, respectively, before the next dispatch. However, if instead more demand arrivals occur, which takes the demand level to 7, she is forced to send the four units on hand. For \( D < 3 \), the optimal policy again delays the shipment until another item is produced.

The production decisions are generally monotone. As the demand level at the retailer increases, the supplier tends to turn on the production. As the inventory level at the supplier increases, this tendency is reversed. If the production is already on (off), it is kept on (off) in more states. However, even in production decisions, we observe a non-monotone behavior due to states \((1,9,1)\) and \((1,10,1)\).

We cannot characterize the optimal policy in a simple way due to its non-monotone behavior. In fact, such non-monotone policies are also difficult to implement in real systems. In the next sections, we analyze supply chains in which the retailer shares only the demand probability distribution, along with its parameters, and the
supplier adapts a time-based, a quantity-based or a TQ-based shipment policy.

3. Time-based dispatch policy

In a time-based dispatch policy, the supplier agrees to deliver a shipment that will satisfy all outstanding demands in every $T$ time units. Then, a dispatch cycle, which refers to the elapsed time between two shipments, is constant at $T$. We note that we use dispatch cycle and transportation cycle interchangeably throughout the text. We first describe the process when system adapts a time-based dispatch policy (Section 3.1). We compute the expected cost in a dispatch cycle in Section 3.2. Finally, Section 3.3 presents the MDP model of this system, which provides the means to evaluate different system choices.

3.1. Description of the system

A time-based policy ensures that the total accumulated demand at the retailer will always be satisfied at a pre-determined shipment date. Hence, the customers exactly know when their orders will be delivered. Moreover, in practical applications, it may be easier to schedule dispatches so that a shipment is realized on a periodic basis. In the transportation contracts between the supply chain members, time-based shipment policies are also known as time-definite delivery agreements. This kind of contracts is

![Fig. 1. Optimal shipment and production policies: (a) $Q$ for $j = 0$, (b) $Q$ for $j = 1$, (c) $j$ for $j = 0$, and (d) $j$ for $j = 1$.](image-url)
common between the manufacturers and their third-party logistics service provider partners.

The drawback of time-based policies stems from the fact that the dispatch quantity is a random variable when the demand is stochastic, so that the dispatch load may not realize the economies of scale for some demand quantities. Furthermore, outsourcing should be possible in order to satisfy all the accumulated orders in a cycle, since it may not be feasible to produce enough items to match the total demand. In the cases when outsourcing is not allowed, the outsourcing cost, \( s \), can be interpreted differently: In certain settings, especially under time-based or quantity-based dispatch policies where the supplier promises to deliver all the accumulated demand at a pre-determined time or upon the accumulation of a pre-determined amount of orders, it is reasonable to assume that the orders that are left unsatisfied after a shipment will be lost and there will be a penalty associated with it. Then, the outsourcing cost, \( s \), can be interpreted as the cost of losing the demand that cannot be satisfied upon the next delivery. This interpretation allows to always have the option of outsourcing. We note that outsourcing (or losing a part of the demand) can be discouraged strongly, by choosing a high enough \( s \).

Fig. 2 shows a typical time evolution of the inventory level and accumulated demand under a time-based policy. In every \( T \) time periods, the supplier dispatches the total demand accumulated during \( T \) time units. In this model, we use the classical produce-up-to inventory model which produces up to a base stock level generally known as \( S \)-policy. According to this policy, when production is started, it continues until a certain level of inventory is reached in order to balance the production setup and inventory holding cost. We let \( Q_{\text{max}} \) denote this maximum level of inventory to stop the production. In the beginning of the process, the supplier had \( I \) products in the inventory and the production process was off. The supplier waited for \( t_I \) time units before she started the production, which continued until there are \( Q_{\text{max}} \) units in the inventory. After that epoch, the supplier does not produce until the dispatch cycle is over. The supplier decides to start production again in the next replenishment cycle. Note that we set \( t_I = T \) if the supplier would not produce anything in that transportation cycle. We note that \( t_I \) is defined only when the production is off because when the production is on, it should continue until \( Q_{\text{max}} \) items are accumulated in the inventory and the production cannot be stopped and restarted without a setup cost.

The supplier knows the probability distribution of the demand with all its parameters, but she can no longer trace the number of outstanding orders. Hence, the state of the system is represented by the pair \((j, l)\), where both \( j \) and \( l \) are defined as in the previous section. Let \( S \) be the state space of the system under a time-based policy, so that \( S = \{ (j, l) : j \in (0,1), I \geq 0 \} \).

We aim to minimize the long-run average cost over an infinite horizon when the system adapts a time-based policy. For this purpose, we optimize the length of the dispatch cycle, \( T \), the maximal inventory level, \( Q_{\text{max}} \), which determines the time to stop production, and \( t_I \)'s, which indicate the time to start production when the cycle begins with \( I \) items in inventory. The first ingredient in this optimization procedure is to calculate the expected cost incurred over a dispatch cycle for given values of \((T, Q_{\text{max}}, t_I)\) (Section 3.2). Next, we calculate the total expected cost per unit time for any given pair of \((Q_{\text{max}}, T)\) through an MDP model, which also determines the corresponding optimal \( t_I \)'s values (Section 3.3). Finally, we perform a line search in two dimensions to find the optimal \((Q_{\text{max}}, T)\) by searching for all possible combinations with increments of 1 for \( Q_{\text{max}} \) and 0.1 for \( T \) up to a safe upper bound for these values.

3.2. Computation of the cost over a dispatch cycle

The total cost in each cycle has three main sources, in addition to the set-up cost of production and transportation: customer waiting cost, outsourcing cost and holding cost. This section presents expressions for each of these cost components.

The expected waiting cost over a dispatch cycle, denoted by \( W^j \), is independent of the state of the system, since all orders are
satisfied in the beginning of each dispatch cycle. Let \( D \) denote the demand accumulated at the retailer during a dispatch cycle with a length of \( T \) time periods. Since the demand arrivals form a Poisson process, it is straightforward to find the expected total waiting cost of accumulated demand during \( T \) time units as:

\[
W^T = \frac{ED[T_w]}{2}.
\] (3.1)

Now, we compute the expected outsourcing cost due to the lack of sufficient products on hand. Let the system start a cycle in state \((j,I)\). We define three relevant random quantities. Consider the case when there is an ongoing production process, so \( j = 1 \). We let \( X_i \) be the quantity of the products to be produced during \( T \) time units if the production process is always on. Similarly, we define \( X_0 \) for \( j = 0 \): In this case, the supplier starts the production after \( t_i \) time units. We let \( X_0 \) be the total number of items to be produced if the production goes on for the whole \( T - t_i \) time units. Since producing one item requires an exponential amount of time with rate \( \mu \), \( X_1 \) and \( X_0 \) are Poisson random variables with means \( T\mu \) and \((T - t_i)\mu\), respectively. Finally, we define \( X \) as the actual amount of production in a dispatch cycle. Because the production is stopped when the inventory level reaches \( Q_{\text{max}} \), we have \( X = \min\{Q_{\text{max}} - I, X_i\} \) when the initial state is \((j,I)\). In order to find the amount of items to be obtained from the supplier, we need the inventory level at the end of the cycle, which is given by \( I + X \). Then the expected outsourcing cost due to the lack of sufficient products on hand is given by:

\[
O^T(j, I) = E( D - I - X ).
\] (3.2)

We note that \( O^T(j, I) \) depends on \( j \) through \( X \), since \( X = \min\{Q_{\text{max}} - I, X_i\} \).

Computing the expected holding cost is more complicated, so we place the details of deriving the total expected holding cost incurred over \( T \) time units in the appendix. Here, we only present its final expression:

\[
H^T(j, I) = \frac{D}{2} [Q_{\text{max}} - I + \min\{Q_{\text{max}} - I, X_i\}] - \frac{1}{2} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} [Q_{\text{max}} - I + \min\{Q_{\text{max}} - I, X_i\}] + \frac{1}{2} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} [Q_{\text{max}} - I + \min\{Q_{\text{max}} - I, X_i\}] - Q_{\text{max}}.
\] (3.3)

3.3. The MDP model of the system

In the MDP model for a time-based policy, the system state is observed at every \( T \) time units, right after a shipment is made. This is in contrast with the full-information model, where the system state is checked at every potential transition epoch. Our aim is to derive the expected total cost incurred in a typical dispatch cycle, and determine the optimal \( t_i \) values, when the system starts in state \((j,I)\) and follows a time-based policy with parameters \((Q_{\text{max}},T)\).

We let \( g^T \) be the expected minimal long-run average cost of the system, which employs a time-based dispatch policy, and \( V^T(j, I) \) be the corresponding relative value function. According to our assumption for this model, whenever the production starts, the supplier must produce till the inventory level reaches \( Q_{\text{max}} \). This assumption determines when to stop the production once it is started. Therefore, when the system is in state \((1,I)\), the supplier does not have a further decision. Then, the Bellman equations of the corresponding MDP in state \((1,I)\) for any given \( T \) and \( Q_{\text{max}} \) are given as follows:

\[
g^T + V^T(1, I) = W^T + O^T(1, I) + H^T(1, I) + E[K_t]
\]

\[
+ E\left[ V^T(i(I + X_1 < Q_{\text{max}}), (I + X - D)^+) \right],
\] (3.4)

where \( X_1 \) is the quantity of the products to be produced during \( T \) time units if the production process is always on. The transportation cost, \( K_t \), depends on the shipped quantity, thus we take the expected value of this cost considering the demand during \( T \) time units. The production is stopped, when the inventory level at the supplier exceeds \( Q_{\text{max}} \), so that the state of the production takes the value of the indicator function \( 1(I + X_1 < Q_{\text{max}}) \), which is set to 1 when \( I + X_1 < Q_{\text{max}} \), and to 0 otherwise.

When the system is in state \((0,I)\), the supplier decides when to start the production. The time till production, \( t_i \), depends on \( I \), and the MDP model solves for optimal \( t_i \)’s for a given pair \((Q_{\text{max}},T)\). The relative value function for state \((0,I)\) is as follows:

\[
g^T + V^T(0, I) = W^T + K_p(D)
\]

\[
+ \min_{0 \leq t \leq T} \left[ O^T(0, I) + H^T(0, I) + K_p(I) \right]
\]

\[
< T + E[V((I + X_0 < Q_{\text{max}}) \wedge (t_i
\]

\[
< T)), (I + X - D)^+)]].
\] (3.5)

where \( X_0 \) is the total number of items to be produced if the production goes on for the whole \( T - t_i \) time units. The production starts if \( t_i < T \), upon which the set-up cost of \( K_p \) is incurred. Furthermore, the production state changes from 0 to 1, only when the production is started in the cycle so that \( t_i < T \), and \( Q_{\text{max}} \) is not reached during the cycle, i.e., \( I + X_0 < Q_{\text{max}} \). The indicator function takes the value of 1, if both of these conditions are satisfied; it is set to 0 otherwise.

We note that the MDP model for a time-based policy has an uncountable action space (for states with \( j = 0 \)), as opposed to the MDP representation of the full-information model. Since the action space is compact, i.e., the set \( 0 \leq t_i \leq T \) is closed and bounded, the solution to this MDP model also exists (see Hernández-Lerma & Lasserre, 1999).

4. Quantity-based dispatch policy

This section considers a quantity-based dispatch policy in which a shipment is made whenever the outstanding customer orders are accumulated to a quantity, \( Q \) (to be optimized). Under a quantity-based policy, the supplier does not trace the level of accumulated demands as in a time-based policy. Hence, the analysis of the quantity-based dispatch policy is very similar to that of the time-based policy. Section 4.1 describes the system that uses a quantity-based dispatch policy. The associated cost in a dispatch cycle is computed in Section 4.2 and a corresponding MDP model is used to evaluate different system choices in Section 4.3.

4.1. Description of the system

In this system, the manufacturer commits to satisfy all the orders at the retailer when the amount of orders reach to a certain level \( Q \). Hence, the supplier needs to decide on the dispatch quantity, \( Q \), in order to minimize total cost. However, now dispatch times are random, which may create a serious drawback in terms of satisfying customer service requirements since the customers will not know exactly when their orders will be delivered. Therefore, similarly to the time-based policy, either the outsourcing option should be always available, or the outsourcing cost should be interpreted as the cost of losing one unit of demand when the total accumulated demand cannot be satisfied upon next delivery.

A typical time evolution of the inventory level and customer orders under a quantity-based policy is presented in Fig. 3. The demand starts accumulating in the retailer. Whenever the accumulated demand reaches the level of \( Q \), the supplier must dispatch a quantity of \( Q \) immediately. When the production process is off and a dispatch cycle starts with an inventory level of \( I \), the supplier decides to turn on the production after \( t_i \) time units. In Fig. 3, the supplier starts the production after \( t_i \) time units in the first
transportation cycle and she continues the production till the inventory level of $Q_{\text{max}}$ is reached. Under this policy, the time length between successive dispatches, $T$, is a random variable, which follows an Erlang distribution with parameters $Q$ and $\lambda$, since a shipment is made when exactly $Q$ demands arrive at the retailer.

In this model, the state of the system is $(j,l)$, and the state space is $\mathbb{S}$. A quantity-based policy aims to minimize the long-run average cost over an infinite horizon by optimizing the parameters, dispatch quantity, $Q$, maximal inventory level $Q_{\text{max}}$, and $t_i$’s. The values of optimal $t_i$’s for any given pair of $(Q_{\text{max}}, Q)$ are computed by an MDP model, and a two-dimensional line search is performed to optimize $(Q_{\text{max}}, Q)$ by considering all possible combinations up to safe upper bound for both values.

### 4.2. Computation of the cost over a dispatch cycle

We first compute the expected waiting cost over a dispatch cycle, denoted by $W^0$. This computation is the same as in Section 3. Only now the time till the next shipment $T$ is random, as opposed to constant, whereas the number of demand arrivals during $T$ time units is constant at $Q$, as opposed to the random demand quantity, $D$. Hence, we have:

$$W^0 = \frac{(Q - 1)E[T]W}{2}. \quad (4.1)$$

To compute the expected outsourcing and holding cost, we define $Y_j$ as the number of items produced in a transportation cycle if the production process was not stopped during that particular cycle. Then $Y_j$ is the quantity of products to be produced until the inventory level reaches $Q_{\text{max}}$ we have $Y = \min\{Q_{\text{max}} - I, Y_j\}$. Now the expected outsourcing cost due to the lack of sufficient products on hand is computed as:

$$O^S(j, l) = E[s(Q - I - Y)]^+. \quad (4.2)$$

The expected holding cost can be computed as in Section 3.2, resulting in the following expression:

$$H^S(j, l) = IE[T]h + \frac{h}{2}E\left[\frac{(T - t_i(1 - j))Y(2Y_j - Y + 1)}{(Y_j + 1)}\right]. \quad (4.3)$$

### 4.3. The MDP model of the system

In this model, we check the system state right after exactly $Q$ orders are dispatched from the supplier to the retailer. We can present the corresponding MDP model for a given pair $(Q_{\text{max}}, Q)$. We define $g^S$ as the expected minimal long-run average cost of the system under a quantity-based dispatch policy, and $V^S(j, l)$ as the corresponding relative value function. Then, the Bellman equations of the corresponding MDP in state $(1, l)$ for any given $Q$ and $Q_{\text{max}}$ are given as follows:

$$g^S + V^S(1, l) = W^S + O^S(1, l) + H^S(1, l) + K_l + E[V^S(l(1 + Y_1 < Q_{\text{max}}), I + Y - D)^+)]. \quad (4.4)$$

The production is stopped when the number of items produced exceeds $Q_{\text{max}}$, which is represented by the indicator function $1(l + Y_1 < Q_{\text{max}})$. When the system is in state $(0, l)$, the supplier needs to decide when to start the production. The MDP model solves for the optimal waiting times until the production starts, $t_i$, when the inventory level is $I$. We let $T$ be large enough, so that all reasonable $t_i$’s should lie below $T$. This ensures that the action space is compact, so that a solution to the MDP model exists (see Hernandez-Lerma
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& Lasserre, 1999). Then, the relative value function for state \((0, I)\) is as follows:

\[
g^{\mathcal{Q}} + V^{\mathcal{Q}}(0, I) = W^{\mathcal{Q}} + K_{r} + \min_{0 \leq t < T} \{O^{\mathcal{Q}}(0, I) + H^{\mathcal{Q}}(0, I) + K_{p} E\{I(t_{1} < T)\}
+ E[V\{((I + Y_{0}) < Q_{\max}) \land (t_{1} < T), (I + Y - D)^{+})\}].
\]

(4.5)

Since \(T\) is a random variable, we take the expectation of the indicator function \(I(t_{1} < T)\), which shows whether the production starts in the current dispatch cycle or not.

5. TQ-based dispatch policy

In this section, we integrate time and quantity-based dispatch policies such that a shipment is made when \(Q\) units of customer orders are accumulated at the retailer or \(T\) time units have passed from the previous shipment, whichever occurs earlier. With this policy, it will be guaranteed that the customer orders will always be delivered by no later than \(T\) time units while taking advantage of the cost effectiveness of the quantity based policy. The analysis of this policy is straightforward since it is a hybrid of time and quantity-based dispatch policies. Hence, we present the results concisely.

In this case, let \(D_{T}\) denote the customer orders accumulated in \(T\) time units. Then, if \(D_{T} < Q\), a shipment is due to the time constraint and if \(D_{T} \geq Q\), a shipment is made with \(Q\) units before \(T\). Similar to the previous sections, we let \(Z\) denote the amount of production in a cycle and we calculate the costs of waiting, outsourcing and holding by using the conditional values depending on which event occurs first, respectively:

\[
W^{\mathcal{Q}} = P(D_{T} < Q) \frac{E[D_{T}]I(D_{T} < Q)Tw}{2} + P(D_{T} \geq Q) \frac{(Q - 1)E[T]\Delta T}{2} \geq Q|w.
\]

\[
O^{\mathcal{Q}}(j, I) = P(D_{T} < Q) E[S(D_{T} - I - Z)^{+} | D_{T} < Q]
\]

\[
+ P(D_{T} \geq Q) E[S(Q - I - Z)^{+} | D_{T} \geq Q];
\]

\[
H^{\mathcal{Q}}(j, I) = P(D_{T} < Q) \left\{ \frac{h}{2} \left[ \frac{(T - t_{1}Z(2Z_{j} - Z + 1) + 1)}{(Z_{j} + 1)} | D_{T} < Q \right] \right\}
\]

\[
+ P(D_{T} \geq Q) \left[ \frac{h}{2} \left[ \frac{(T - t_{1}Z(2Z_{j} - Z + 1) + 1)}{(Z_{j} + 1)} | D_{T} \geq Q \right] \right\}.
\]

Then, the Bellman equations of the corresponding MDMP model in states \((1, I)\) and \((0, I)\) for any given \(T\), \(Q\) and \(Q_{\max}\) are as follows:

\[
g^{T, Q} + V^{T, Q}(1, I) = W^{T, Q} + O^{T, Q}(1, I) + H^{T, Q}(1, I) + E[K_{1}]
\]

\[
+ E\left[ V^{T, Q}(I + Z_{1} < Q_{\max}), (I + Z - D)^{+}) \right],
\]

\[
g^{T, Q} + V^{T, Q}(0, I) = W^{T, Q} + E[K_{1}] + \min_{0 \leq t_{1} < T} \{O^{T, Q}(0, I) + H^{T, Q}(0, I)
\]

\[
+ K_{p} E\{I(t_{1} < \min(T, T_{Q}))\} + E[V\{(I + Z_{0} < Q_{\max})
\land (t_{1} < \min(T, T_{Q})), (I + Z - D)^{+})\}].
\]

6. Numerical analysis

In this section, we present the results of our numerical study to compare the performances of the optimal time-based, the quantity-based and TQ-based models with respect to the optimal solutions of the full-information model, which serves as a lower bound on the cost of the system. In the full-information model, the optimal policy is used with a complete knowledge on the system state. However, other policies use partial information on the system state, and provide heuristic solutions. We note that modeling the optimal policy with partial information is not possible as the corresponding actions cannot be defined properly without a given policy structure. Thus, the comparisons of the full-information model with the other policies involve the effects of the full information as well as following the optimal policy for this case. This section also numerically analyzes the characteristics of the optimal solutions for the heuristic policies and the impacts of various factors, such as production and demand rates, and cost figures, on these four models.

The base case of our numerical study uses the parameters \(\lambda = 0.3, \mu = 0.7, K_{p} = 200, P = 5, R = S = 50\) such that \(K_{p} = 50 + 50Q/5\), \(h = 1\), \(w = 10\) and \(s = 350\), which allows us to compare our results with those of Çetinkaya et al. (2006). Table 1 presents the cost performances of the full-information model, the time-based model and the quantity-based model for different parameter settings, e.g. \(g^{T}\) and \(g^{Q}\), respectively. Implementation of TQ-based model with the objective of cost minimization sets the optimal shipment time to infinity, i.e., \(T = \infty\). Hence, the results of the TQ-based model are exactly the same as the results in the quantity-based model. The table also gives the percentage of cost increases when the time-based and quantity-based policies are implemented instead of the optimal policy. Finally, it includes the optimal shipment times, denoted by \(T^{*}\), and the optimal shipment amounts, denoted by \(Q^{*}\), as well as the optimal maximum inventory levels, denoted by \(Q_{\max}\), and \(Q_{\max}^{T}\), in the time-based and quantity-based models, respectively. The first row of Table 1 shows the results with the base case parameter. We carry on our sensitivity analysis with respect to the parameters, \(h, w, \lambda, \mu, K_{p}, R\) and \(s\), by letting them take six different values including the base value, and report all our findings in Table 1. Our models do not assume any specific relations between the cost values. However, since the holding and waiting cost are incurred per unit per time unit, they are expected to be lower than the other cost values. We note that the time scale is the same in all examples, since we keep \(\lambda + \mu = 1\) even when the ratio of \(\lambda/\mu\) varies.

6.1. Comparison of the performances of the models

The full-information model has two major benefits over the time-based and quantity-based policies: In the full-information model, the supplier (1) has continuous access to the demand information, and (2) does not commit herself to satisfy the demand at certain time intervals or whenever the demand accumulates to a certain level. Accordingly, the long-run average cost of the full-information model are, in general, significantly lower than those of the other two policies. Still, the performance of the quantity-based policy is quite acceptable in most cases. More explicitly, the cost increase in the quantity-based model is less than 5%, 10% and 15% in 10, 22, and 26 instances out of a total of 31 instances, respectively. The time-based policy, on the other hand, always performs as the worst of the three policies. As discussed earlier, the time-based policy has an additional disadvantage due to the random dispatch quantities, which frequently violates the economies of scale. Hence, it is not surprising that the long-run average cost increases at least 10% in 30 instances, when the time-based policy is implemented instead of the full-information model policy.

In addition to the results in Table 1, we also consider the hybrid quantity-and-time-based model and we observe that, with the objective of cost minimization, the optimal time limit for shipments happens to be \(T = \infty\) and the results with the hybrid model are exactly the same as the results in the quantity based model. Similar to the observations in Mutlu et al. (2010), we also observe that even though the hybrid model does not add any value over the quantity-based model in terms of cost minimization, the expected maximum waiting time of the customers (i.e. expected time between two shipments) decreases with the hybrid policy compared to the quantity-based policy. In Fig. 4, we present the percent
differences of cost and expected maximum waiting time between the hybrid policy and the quantity-based policy for given \( T \) values. We observe that as \( T \) increases cost of the hybrid policy approaches to the cost of the quantity-based policy but the savings obtained by the hybrid policy on the expected maximum waiting time also decreases with \( T \). We observe that the waiting times of the customers can be decreased substantially in exchange for a small increase in cost by a suitable choice of \( T \). For example when \( T = 10 \), the expected maximum waiting time of the customers are almost 20% less with the hybrid policy compared to the quantity-based policy, while the cost only increases by about 2.5%.

### 6.2. The effects of parameters on the performances of the models

Fig. 5 presents the long-run average cost values for the three models under different parameter settings. As expected, the long-run average cost of all models increases in all the cost parameters, as well as in the load of the system, \( k/l \). When the load of the system is higher, the system is more vulnerable to the random fluctuations in the demand and production, which increase the cost. The relative performances of both quantity-based and time-based policies worsen, as the inventory holding cost, \( h \), increases and/or the customer waiting cost, \( w \), decreases. As the inventory

**Table 1** Summary of numerical results. The parameters of the base case are as follows: \( \lambda = 0.3, \mu = 0.7, K_p = 200, P = 5, R = 50 \) such that \( K_t = 50 + 50(Q/5), \) \( h = 1, w = 10 \) and \( s = 350. \)

<table>
<thead>
<tr>
<th>Parameters varied</th>
<th>FI model</th>
<th>Time-based policy</th>
<th>Quantity-based policy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( g )</td>
<td>( T^f )</td>
<td>( Q_{max}^f )</td>
</tr>
<tr>
<td>Base case</td>
<td>28.7</td>
<td>5.1</td>
<td>12</td>
</tr>
<tr>
<td>( h = 0.25 )</td>
<td>24.41</td>
<td>5.8</td>
<td>21</td>
</tr>
<tr>
<td>( h = 0.5 )</td>
<td>26.2</td>
<td>5.7</td>
<td>16</td>
</tr>
<tr>
<td>( h = 2 )</td>
<td>31.94</td>
<td>4.6</td>
<td>9</td>
</tr>
<tr>
<td>( h = 4 )</td>
<td>35.89</td>
<td>3.7</td>
<td>7</td>
</tr>
<tr>
<td>( h = 8 )</td>
<td>40.88</td>
<td>2.8</td>
<td>5</td>
</tr>
<tr>
<td>( w = 3 )</td>
<td>14.42</td>
<td>13.4</td>
<td>16</td>
</tr>
<tr>
<td>( w = 2 )</td>
<td>17.54</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>( w = 5 )</td>
<td>23.23</td>
<td>8.4</td>
<td>13</td>
</tr>
<tr>
<td>( w = 20 )</td>
<td>34.74</td>
<td>2.1</td>
<td>12</td>
</tr>
<tr>
<td>( w = 50 )</td>
<td>40.24</td>
<td>1.1</td>
<td>10</td>
</tr>
<tr>
<td>( \lambda/\mu = 0.1 )</td>
<td>14.03</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>( \lambda/\mu = 0.3 )</td>
<td>24.5</td>
<td>4.9</td>
<td>10</td>
</tr>
<tr>
<td>( \lambda/\mu = 0.5 )</td>
<td>30.61</td>
<td>5.3</td>
<td>13</td>
</tr>
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<td>( \lambda/\mu = 0.7 )</td>
<td>34.97</td>
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<td>( \lambda/\mu = 0.9 )</td>
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<tr>
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<td>( K_p = 100 )</td>
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<td>10</td>
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<tr>
<td>( K_p = 400 )</td>
<td>31.76</td>
<td>5.4</td>
<td>15</td>
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<tr>
<td>( K_p = 800 )</td>
<td>36.13</td>
<td>5.4</td>
<td>20</td>
</tr>
<tr>
<td>( K_p = 1600 )</td>
<td>42.94</td>
<td>5.4</td>
<td>26</td>
</tr>
<tr>
<td>( R = 0 )</td>
<td>21.43</td>
<td>3.3</td>
<td>12</td>
</tr>
<tr>
<td>( R = 150 )</td>
<td>39</td>
<td>9.2</td>
<td>14</td>
</tr>
<tr>
<td>( R = 300 )</td>
<td>49.52</td>
<td>13.8</td>
<td>16</td>
</tr>
<tr>
<td>( R = 600 )</td>
<td>65.14</td>
<td>16.2</td>
<td>17</td>
</tr>
<tr>
<td>( R = 1200 )</td>
<td>92.69</td>
<td>27.6</td>
<td>20</td>
</tr>
<tr>
<td>( s = 50 )</td>
<td>28.45</td>
<td>5.3</td>
<td>10</td>
</tr>
<tr>
<td>( s = 100 )</td>
<td>28.67</td>
<td>5.3</td>
<td>11</td>
</tr>
<tr>
<td>( s = 200 )</td>
<td>28.69</td>
<td>5.2</td>
<td>12</td>
</tr>
<tr>
<td>( s = 400 )</td>
<td>28.70</td>
<td>4.9</td>
<td>12</td>
</tr>
<tr>
<td>( s = 800 )</td>
<td>28.70</td>
<td>4.9</td>
<td>13</td>
</tr>
</tbody>
</table>

**Fig. 4.** Comparison of the cost and expected maximum waiting time of the hybrid policy vs. the quantity-based policy.
holding cost increases, it becomes more important to better manage the production and the inventory levels at the supplier. The continuous access of the supplier to the demand information in the full-information model provides a crucial benefit over the other policies in systems with higher inventory cost. A decrease in the waiting cost has a similar effect, since it allows the holding cost to become the major cost component. Moreover, the inventory levels under the time-based and quantity-based policies are generally higher than those in the full-information model due to their commitment to fulfill all the accumulated orders at every \( T \) time units and when \( Q \) orders are accumulated, respectively. In both policies, the supplier builds up excess inventory in order to avoid excessive outsourcing cost. In the full-information model, on the other hand, the supplier may allow the orders to wait a little longer until she produces the required amount without needing to source. The commitment of the supplier hurts her more when the inventory cost becomes a major component of the total cost, which happens when \( w \) decreases or \( h \) increases. In fact, when the transportation cost, \( R \), and the production setup cost, \( K_p \), increase, the performances of both quantity-based and time-based policies improve. Finally, the quantity-based policy performs better as the load of the system, i.e., the ratio of the demand rate over the production rate, \( \lambda/\mu \), increases.

### 6.3. The effects of parameters on the decisions of the models

Table 1 shows that the time-based policy leads to slightly higher maximum inventory levels than the quantity-based policy (\( Q_{\text{max}} \) levels). Moreover, further analysis of Table 1 reveals that the average length of the transportation cycles is lower in the time-based policies \((T')\) when compared to that in the quantity-based policies \((Q'/\lambda)\). Hence, the expected shipment frequencies are higher under the time-based policy. Since the amount of accumulated orders in a cycle is random under the time-based policy, the supplier chooses to keep higher inventory levels and to ship more frequently in order to avoid the risk of high customer waiting cost and high outsourcing cost. On the other hand, under the quantity-based policy, the manufacturer knows exactly how many units are needed in that cycle, and thus has a lower risk of outsourcing, which allows her to hold lower inventory levels compared to the time-based policy. Moreover, the quantity-based policy benefits from the economies of scale in transportation. Finally, we observe that, in the quantity-based model the maximum inventory levels are generally a multiple of the shipment amounts, especially when the shipment amounts are high. However, there are several exceptions to this observation since the supplier sometimes chooses to produce some extra items that can be considered as a safety stock against the risk of not being able to produce the required shipment amounts in the first transportation cycle of the next replenishment cycle.

The average lengths of the transportation cycles increase under both the time-based and quantity-based policies, as the customer waiting cost, \( w \), decreases, and the load of the system, \( \lambda/\mu \), or the transportation cost, \( R \), increase. The effects of \( w \) and \( R \) are obvious. As for the system load, when the demand arrival rate, \( \lambda \), is closer to the production rate, \( \mu \), it takes longer to produce the items to fulfill the demand, so that the transportation cycles lengthen. As the shipment frequencies decrease, the quantity dispatched in each shipment increases, which, generally, increases the maximum inventory levels, \( Q_{\text{max}} \) and \( Q'_{\text{max}} \). The maximum inventory levels are also increasing in \( K_p \) and \( s \), and decreasing in \( h \). The effects of the outsourcing cost, \( s \), and the holding cost, \( h \), are clear. When the production set-up cost of the supplier, \( K_p \), increases, the supplier produces more during a replenishment cycle, which leads to higher maximum inventory levels.

The truck capacity \( P \) affects the shipment quantities by forcing full-truck load shipments instead of shipping a little more or less than the capacity in certain cases and this force gets stronger as the cost of each truck, \( S \), increases. However, full-truck loads are not always enforced due to the effects of the other parameters such as...
6.4. Managerial insights

The full-information model performs better than all the heuristic policies. However, as discussed earlier, the implementation of the full-information model is difficult in some situations since it requires a close interaction between the retailer and the supplier. Thus, the heuristic policies may remain as the only viable options in many supply chains. Similar to the results in Çetinkaya et al. (2006), we suggest the implementation of quantity-based policies, especially when \(w, \sqrt{\mu}, K,\) and \(k_p\) are high and \(h\) and \(s\) are low. Çetinkaya et al. (2006) state that in practice, typically, quantity-based policies are used for higher volume (high \(k\)), and lower value (low \(h\)) items, such as peripheral computer equipment, which is consistent with our results. However, we note that, especially for low values of \(w\) and high values of \(h\), the quantity-based policy might lead to poor results, causing a cost increase of more than 30\% when compared to the full-information solution. This suggests that it may be worth pushing for the full-information sharing in these cases, rather than using the heuristic policies. In addition, if \(\sqrt{\mu}, K,\) and \(k_p\) are low or if \(s\) is high in a system, then the full-information model might lead to significant savings.

In practice, we observe that both time-based and quantity-based policies are commonly used. Even though the time-based policy performs worse than the quantity-based policy, it is easier to implement a time-based dispatch policy than a quantity-based policy since the time-based policy allows periodical shipments, which facilitates the timely planning of the shipments. In addition, with the time-based policy, the customers will know exactly when they will obtain their products. Thus, practical applications value the simplicity and periodic delivery advantages of the time-based dispatch policies, rather than the resulting cost. Finally, TQ-based policies provide a good alternative when the maximal shipment time, \(T\), is determined according to the customer relations. These policies can decrease the waiting time of the customers significantly in exchange for a small increase in cost.

7. Conclusion

In this study, we analyze the integrated production and dispatch policies in supply chains in which the retailer does not hold any inventory but accumulates the customer orders to satisfy at a later time. We consider a full-information model first and observe that the optimal solution is non-monotonic and complex, thus hard to implement. Moreover, this full-information model requires a continuous flow of information between the retailer and the supplier, which may be problematic in some cases. Hence, we also analyze time-based, quantity-based and TQ-based dispatch policies which are easier to implement. We observe that the quantity-based policy is superior to the time-based dispatch policy in terms of the resulting supply chain cost. However, in certain cases even the quantity-based policy leads more than 30\% increase in the total cost, when compared to the full-information model. Thus, in such cases, trying to implement the optimal policy of the full-information model might be worth the effort instead of using the simpler policies.

The common use of the time-based policy in practice brings a natural question as to why the industry keeps using this policy, although it leads to significant cost increases. The high customer satisfaction under time-based policies may be a key to understand this phenomena. Hence, further research to quantify the customer satisfaction may lead to better models, which represent a bigger picture in this context. At this point, we should note that TQ-based dispatch policies provide a strong alternative. Even though the performance of the optimal TQ-based dispatch policy is exactly the same as the quantity-based dispatch policy, the waiting time of the customers can be decreased in expense of a small increase in cost.

Appendix A

Let the system start in state \((1, I)\), so that the production process is on in the beginning of the dispatch cycle. The total inventory carried in \(T\) time units can be found by computing the area below the curve of the inventory level at the supplier (see Fig. 2). The holding cost have two components. The first component is due to carrying the initial inventory for \(T\) time units, so it is equal to \(Ith\). The second one is due to the items produced during the cycle, which needs a detailed account.

Remember that \(X_t\) is the random amount of items to be produced during \(T\) time units if the production goes on for the whole \(T\) time units, and \(X\) is the random amount of actual production during \(T\) time units. We consider two cases:

**Case 1:** \(I + X_t < Q_{\text{max}}\)

In this case, the production process is on through the whole cycle, and \(X = X_t\). This corresponds to the second dispatch cycle in Fig. 2. The output of the production process is Poisson. Hence, when there are exactly \(X_t\) product completions in \(T\) time units, the completion times are uniformly distributed in \((0, T)\). Moreover, the expected time between two completions is \(T/(X_t + 1)\), since \(X_t\) completions divide \(T\) time units into \(X_t + 1\) intervals. From Fig. 2, we observe that between the first and second completion times, one item is carried. In general, between \(ith\) and \((i + 1)\)st completion times \(i\) items are carried. Therefore:

\[
C_{th}(1, I) = Ih + E\left[\frac{T}{X_t + 1} \cdot \frac{X_t}{X_t + 1} \cdot \frac{(Q_{\text{max}} - I)(Q_{\text{max}} - I + 1)}{2} \right] + \frac{(X_t - (Q_{\text{max}} - I)T)}{X_t + 1} \cdot \frac{(Q_{\text{max}} - I)}{h}.
\]

When \(C_{th}(1, I)\) is written in terms of \(X\), the first two terms of the two cases coincide, leading to:

\[
C_{th}(1, I) = Ih + E\left[\frac{T}{X_t + 1} \cdot \frac{X(X + 1)}{2} + \frac{(X_t - X)T}{X_t + 1} X\right]h.
\]
which is further simplified by straightforward algebra into the following:
\[ C_j(I, J) = MH + \frac{Th}{2} E \left[ \frac{X(2X_j - X + 1)}{(X_j + 1)} \right]. \]

When \( j = 0 \), the dispatch cycle begins while the production is turned off. The supplier decides to start the production after \( t_I \) time units. If \( t_I = T \), then there will be no production in that cycle, so that the inventory level will be constant at \( I \) through the whole cycle. Otherwise, we use \( X_0 \), which is defined previously as the total number of items to be produced if the production goes on for the whole \( T - t_I \) time units. Then, the actual amount of production, \( X \), is given by \( X = \min(X_0, Q_{\text{max}} - I) \). Now, finding the total holding cost is very similar to the case with \( j = 1 \), where we only need to replace \( T \) by \( T - t_I \) and \( X_0 \) by \( X_0 \). Therefore, the following expression is valid for both \( j \):
\[ C_j(I, I) = MH + \frac{(T - t_I(1 - j))th}{2} E \left[ \frac{X(2X_j - X + 1)}{(X_j + 1)} \right]. \]

where \( X = \min(X_0, Q_{\text{max}} - I) \).

References


