Coloring, Sparseness, and Girth

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(joint work with Noga Alon, Alexandr V. Kostochka, Benjamin Reiniger, and Xuding Zhu)

A *proper coloring* of a graph G assigns colors to its vertices so that adjacent vertices receive distinct colors. The *chromatic number* of G is the least k such that G has a proper coloring from a set of k colors. A *list assignment* L on G assigns a list $L(v)$ of available colors to each vertex v. An *L-coloring* is a proper coloring with the color on each vertex chosen from its list. A graph is *k-choosable* if it is L-colorable whenever each list in the assignment L has size at least k. The lists could be identical, so the least k such that G is k-choosable is at least the chromatic number.

We construct existence and sharpness examples for several questions in coloring and list coloring, using sparse graphs constructed from very tall trees. An *r-augmented tree* consists of a rooted tree plus edges added from each leaf to r ancestors. For $d, g, r \in \mathbb{N}$, we construct a bipartite r-augmented complete d-ary tree having girth at least g, called a *(d, r, g)-graph*. The height of such trees must grow extremely rapidly in terms of the girth.

We give several applications of (d, r, g)-graphs, producing the following: (1) A new simple construction of graphs (and uniform hypergraphs) with large girth and chromatic number. (2) Construction of bipartite graphs with large girth that are not k-choosable even though all proper subgraphs have average degree at most $2(k-1)$ (maximum average degree at most $2(k-1)$ makes a bipartite graph k-choosable). (3) Construction of a bipartite graph with large girth having a k-uniform list assignment L from which no proper coloring can be chosen even though the lists at adjacent vertices have only one common element (having two common elements guarantees L-colorability). (4) Enhancement of (2) so that the union of the lists has size $2k-1$ (size at most $2k-2$ guarantees L-colorability).

MSC2000: 05C15, 05C05.

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