Question 1: A steel machine part is statically loaded and has a yield strength of 320 MPa. For each of the following stress states find the factor of safety using each of the three static failure theories.

a) $\sigma_x = 60$ MPa $\qquad \sigma_y = -30$ MPa $\qquad \sigma_z = -20$ MPa $\qquad \tau_{xy} = 40$ MPa

b) $\sigma_x = 70$ MPa $\qquad \tau_{xy} = 30$ MPa

c) $\sigma(MPa) := \begin{pmatrix} -40 & 30 & 0 \\ 30 & -60 & 0 \\ 0 & 0 & -10 \end{pmatrix}$

Solution:

Steel is a ductile material so we will use the ductile static failure theories. First the principal stresses for the given stress state should be calculated. (refer to Tutorial 2 - Question 1)

$$\sigma^3 - \left(\sigma_x + \sigma_y + \sigma_z\right)\cdot\sigma^2 + \left(\sigma_x\sigma_y + \sigma_x\sigma_z + +\sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right)\cdot\sigma$$
$$- \left(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2\right) = 0$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

a) Inserting the known stresses to the given eqn.

$I_1 = 10 \qquad I_2 = -4000 \qquad I_3 = 68000$

$$\sigma^3 - 10\cdot\sigma^2 - 4000\cdot\sigma - 68000 = 0$$

Recall the roots of the equation provides the principal stresses. Solving and arranging;

$\sigma_1 = 75.21$ MPa $\qquad\qquad \sigma_2 = -20$ MPa $\qquad\qquad \sigma_3 = -45.21$ MPa

Factor of safety for each failure theories :

i) <u>Maximum Normal Stress Theory:</u>

(Theory states that failure occurs if any of the principal stresses exceeds the yield strength of the material.)

$$\sigma_{max} = \frac{S_y}{n} \qquad => \qquad n = \frac{320}{75.21} \qquad => \qquad n = 4.26$$

ii) <u>Maximum Shear Stress Theory:</u>

(Theory states that yielding starts whenever the maximum shear stress at any point becomes equal to the maximum shear stress in a tension test specimen of the same material when that specimen starts yielding)

$$\tau_{max} = \frac{S_y}{2n} \qquad\qquad \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{75.21 - (-45.21)}{2} = 60.21 \text{ MPa}$$

$$n = \frac{320}{2 \cdot 60.21} \qquad \Rightarrow \qquad \underline{n = 2.66} \qquad \text{(minimum)}$$

iii) <u>Distortion Energy Theory:</u>

(Theory states that yielding occurs whenever the distortion energy in a unit volume reaches the distortion energy in the same volume corresponding to the yield strength in tension or compression)

the von Mises stress $\quad \sigma' = \left[\dfrac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2} \right]^{1/2} = \dfrac{S_y}{n} \qquad \Rightarrow$

$$n = \frac{320}{\left[\dfrac{(75.21 + 20)^2 + (-20 + 45.21)^2 + (75.21 + 45.21)^2}{2} \right]^{1/2}} \qquad \Rightarrow \qquad n = 2.91$$

b) For the given stress state

$\quad I_1 = 70 \qquad I_2 = -900 \qquad I_3 = 0$

$\quad \sigma^3 - 70 \cdot \sigma^2 - 900 \cdot \sigma = 0$

Solving and arranging;

$\quad \sigma_1 = 81.1 \text{ MPa} \qquad\qquad \sigma_2 = 0 \text{ MPa} \qquad\qquad \sigma_3 = -11.1 \text{ MPa}$

Factor of safety for each failure theories :

i) <u>Maximum Normal Stress Theory:</u>

$$\sigma_{max} = \frac{S_y}{n} \qquad \Rightarrow \qquad n = \frac{320}{81.1} \qquad \Rightarrow \qquad n = 3.95$$

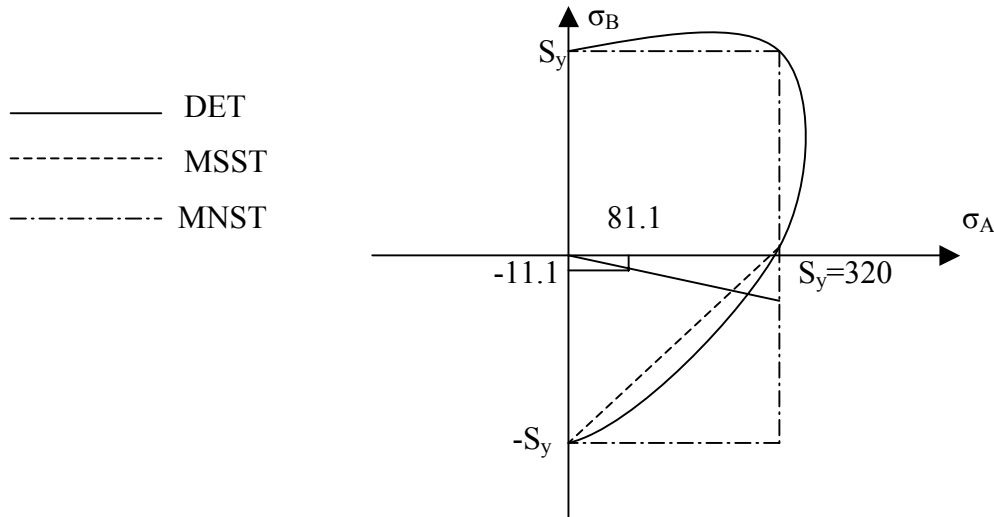ii) <u>Maximum Shear Stress Theory:</u>

$$\tau_{max} = \frac{S_y}{2n} \qquad\qquad \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{81.1 - (-11.1)}{2} = 46.1 \text{ MPa}$$

$$n = \frac{320}{2 \cdot 46.1} \qquad \Rightarrow \qquad \underline{n = 3.47} \qquad \text{(minimum)}$$

iii) <u>Distortion Energy Theory:</u>

$$\sigma' = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2} \right]^{1/2} = \frac{S_y}{n}$$

$$n = \frac{320}{\left[\dfrac{(81.1-0)^2 + (0+11.1)^2 + (81.1+11.1)^2}{2} \right]^{1/2}} \qquad \Rightarrow \qquad n = 3.67$$

Note: The result according to Maximum Normal Stress Theory is a misleading result as the stress state falls into the 4th quadrant in the $\sigma_A - \sigma_B$ graph. The result according to Maximum Shear Stress Theory can be interpreted as the most conservative one whereas the one obtained by Distortion Energy Theory is slightly greater and a more realistic one when compared with experimental results.

c)
$$\sigma := \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$
is the matrix representation of the stress state of an element.

For the given stress state,

$I_1 = -110 \quad I_2 = 2500 \qquad I_3 = -15000$

$\sigma^3 - 110 \cdot \sigma^2 + 2500 \cdot \sigma - 15000 = 0$

Solving and arranging;

$\sigma_1 = -10$ MPa $\qquad\qquad \sigma_2 = -18.4$ MPa $\qquad\qquad \sigma_3 = -81.6$ MPa

Factor of safety for each failure theories :

i) <u>Maximum Normal Stress Theory:</u>

(max. in compression) $\sigma_{max} = \dfrac{S_y(=S_{yc})}{n}$ => $n = \dfrac{-320}{-81.6}$ =>   n = 3.92   (minimum)

ii) <u>Maximum Shear Stress Theory:</u>

$$\tau_{max} = \frac{S_y}{2n} \qquad\qquad \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{-10-(-81.6)}{2} = 35.8 \text{ MPa}$$

$$n = \frac{320}{2 \cdot 35.8} \qquad => \qquad n = 4.47$$

iii) <u>Distortion Energy Theory:</u>

the von Mises stress $\sigma' = \left[\dfrac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2} \right]^{1/2} = \dfrac{S_y}{n}$ =>

$$n = \frac{320}{\left[\dfrac{(-10+18.4)^2 + (-18.4+81.6)^2 + (-81.6+10)^2}{2} \right]^{1/2}} \qquad => \qquad n = 4.72$$

Question 2: A steel LPG tank is shown in the figure. The wall thickness of the tank is 15 mm and has a yield strength of 340 MPa. The full weight of the tank is 6500 kg and the internal pressure is 3 MPa. Calculate the factor of safety of the tank according to the distortion energy theory.
(checking $\dfrac{t}{r} = \dfrac{15}{750} = \dfrac{1}{50} < \dfrac{1}{20}$ the tank can be treated as thin-walled pressure vessel)
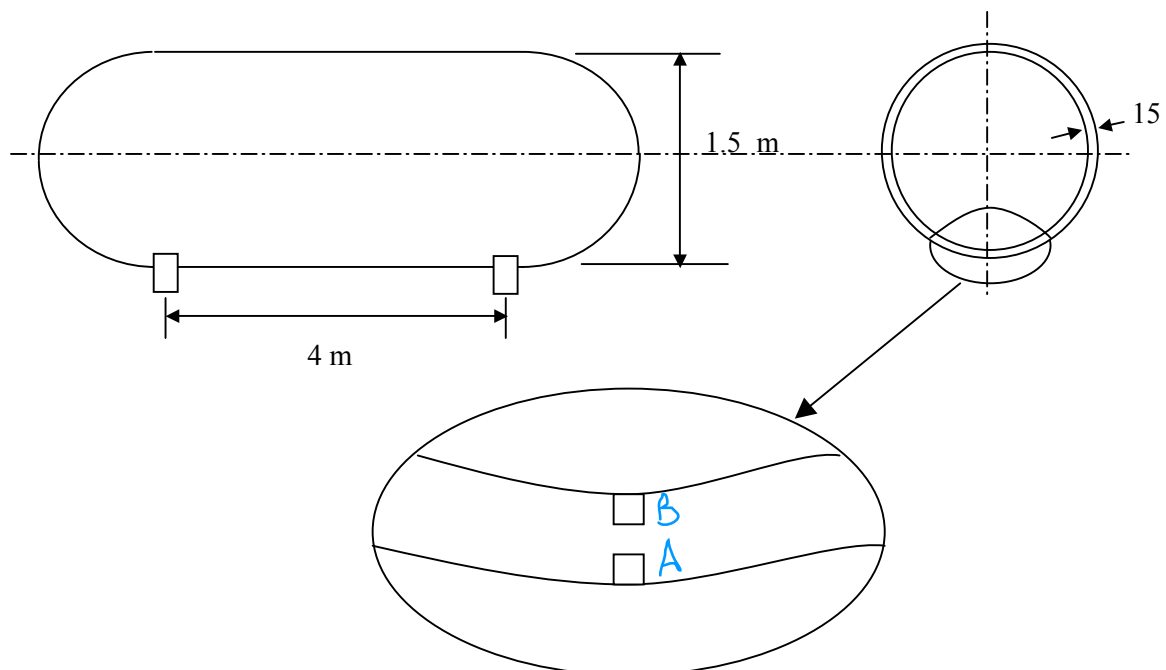
Solution:
First the principal stresses should be calculated for both cylindrical and spherical sections.

For cylindrical vessel:
For point A:

tangential stress $\sigma_t = \dfrac{Pr}{t} = \dfrac{3 \cdot 750}{15} = 150$ MPa   radial stress $\sigma_{r.A} = -p = -3$ MPa

longitudinal stress $\sigma_\ell = \dfrac{Pr}{2t} = \dfrac{3 \cdot 750}{2 \cdot 15} = 75$ MPa

bending stress due to weight of the tank: (consider weight as a concentrated force which is a conservative assumption compared with the distributed weight assumption)

$$I = \frac{\pi}{64}\left(D_o^4 - D_i^4\right) = \frac{\pi}{64}\left(1500^4 - 1470^4\right) = 1.929 \cdot 10^{10} \text{ mm}^4$$

$$\sigma_{b,A} = \frac{Mc}{I} = \frac{63.766 \cdot 10^6 \cdot (750 - 15)}{1.929 \cdot 10^{10}} \cong 2.4 \text{ MPa}$$

At the bottom of the tank the tensile stresses will be larger, so the bottom mid-point is critical. Recalling there will be no traverse shear stress due to weight at the bottom fiber, the axial stresses are to be taken as principal stresses. Arranging as $\sigma_1 > \sigma_2 > \sigma_3$;

$\sigma_1 = \sigma_t = 150$ MPa $\sigma_2 = \sigma_\ell + \sigma_b = 77.5$ MPa $\sigma_3 = \sigma_r = -3$ MPa

After calculating the stress state we can find the factor of safety using the distortion energy
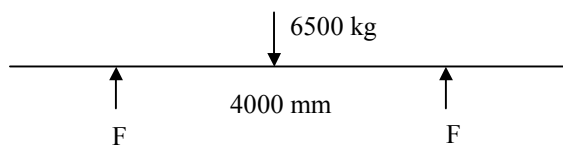
$$F = \frac{6500 \cdot 9.81}{2} = 31883 \text{ N}$$

$$M = 31883 \cdot \frac{4000}{2} = 63.766 \cdot 10^6 \text{ Nmm}$$

theory:

$$\sigma' = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}\right]^{1/2} = \frac{S_y}{n}$$

$$n = \frac{340}{\left[\dfrac{(150 - 77.5)^2 + (77.5 + 3)^2 + (150 + 3)^2}{2}\right]^{1/2}} \qquad => \qquad n = 2.56$$

For point B:
tangential stress $\sigma_t = 150$ MPa   radial stress $\sigma_{r.B} = 0$

longitudinal stress $\sigma_\ell = 75$ MPa

$$\sigma_{b,B} = \frac{Mc}{I} = \frac{63.766 \cdot 10^6 \cdot 750}{1.929 \cdot 10^{10}} \cong 2.5 \text{ Mpa}$$

Comment: Since a thin walled cylinder is used, as it can be seen in above equations, bending moments at point A and B can be considered to be equal. Also, pressure in the tank is small which results in small radial stress at point A compared to longitudinal and tangential stresses. Therefore checking safety factor according to stress element at point A is sufficient for this problem.

For the spherical cap:

on spherical shells stresses in orthogonal directions are same:

$$\sigma_1 = \sigma_2 = \sigma_t = \frac{Pr}{2t} = \frac{3 \cdot 750}{2 \cdot 15} = 75 \text{ MPa}$$

$$\sigma_1 = \sigma_2 = 75 \text{ MPa} \qquad\qquad \sigma_3 = \sigma_r = p = -3 \text{ MPa}$$

$$n = \frac{340}{\left[\dfrac{(75-75)^2 + (75+3)^2 + (75+3)^2}{2}\right]^{1/2}} \qquad\qquad n = 4.36$$

Factor of safety used for the production of the tank is 2.56 (the smaller of the two factors calculated above).

Question 3: A cast iron structure is loaded as shown in the figure. The material has S_{ut} = 325 MPa and S_{uc} = 912 MPa. Find the factor safety of the structure using brittle failure theories at the points A and B (Coulomb-Mohr and Modified Mohr).
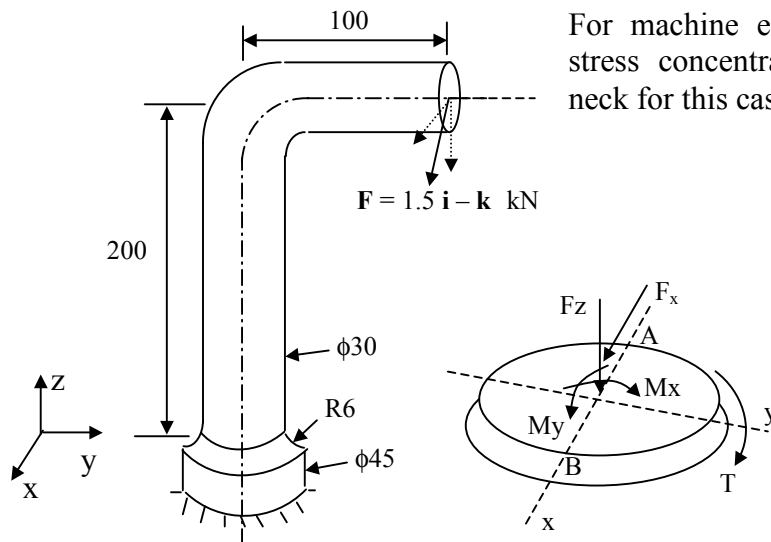
For machine elements made of brittle materials stress concentrations should be considered. The neck for this case is critical.

$$M_x = F_z \cdot 100 = 100000 \text{ N.mm}$$

$$M_y = F_x \cdot 200 = 300000 \text{ N.mm}$$

$$T = F_x \cdot 100 = 150000 \text{ N.mm}$$

$$F_z = 1000 \text{ N}$$

$$F_x = 1500 \text{ N}$$

Solution:
The maximum bending moment on the shoulder is to be calculated using M_y (in N.m).

The transverse shear due to F_x at points A and B is zero.

As stated before, the stress concentrations should be considered on brittle elements. Certain fillets, notches, holes, grooves on the element should be checked as critical sections, as the stress concentrates around these sections.

For $\dfrac{D}{d} = \dfrac{45}{30} = 1.5$ and $\dfrac{r}{d} = \dfrac{6}{30} = 0.2$ $\Rightarrow$ $K_{t.axial} = 1.57$ (Fig E-1, Norton, pp.994)

$K_{ts.torsion} = 1.25$ (Fig E-3) $\qquad\qquad$ $K_{t.bending} = 1.4$ (Fig. E-2)

Stresses at the maximum tension (point A) and compression (point B) points on the critical section, respectively:

$$\sigma_A = K_{t.bend} \cdot \frac{Mc}{I} - K_{t.axial} \cdot \frac{F_z}{A} \quad \text{(tens.+comp.)} \qquad \sigma_B = -K_{t.bend} \cdot \frac{Mc}{I} - K_{t.axial} \cdot \frac{F_z}{A} \quad \text{(comp+comp)}$$

$$\tau_0 = K_{ts.torsion} \cdot \frac{Tc}{J} \quad \text{(same for all points)}$$

$$I = \frac{\pi}{64} D^4 = \frac{\pi}{64} 30^4 = 3.976 \cdot 10^4 \ mm^4 \qquad J = 2 I = 7.952 \cdot 10^4 \ mm^4 \qquad A = \frac{\pi}{4} D^2 = 706.86 \ mm^2$$

At point A: $\quad \sigma_A = 1.4 \cdot \dfrac{300 \cdot 10^3 \cdot 15}{3.976 \cdot 10^4} - 1.57 \cdot \dfrac{1000}{706.86} = 156.2 \ \text{MPa}$

At point B: $\quad \sigma_B = -1.4 \cdot \dfrac{300 \cdot 10^3 \cdot 15}{3.976 \cdot 10^4} - 1.57 \cdot \dfrac{1000}{706.86} = -160.7 \ \text{MPa}$

shear stress $\quad \tau_0 = 1.25 \cdot \dfrac{150 \cdot 10^3 \cdot 15}{7.952 \cdot 10^4} = 35.37 \ \text{MPa}$ (show the direction on the cube below !)
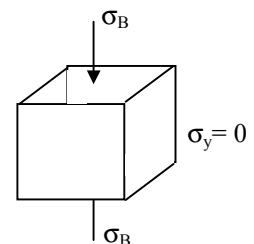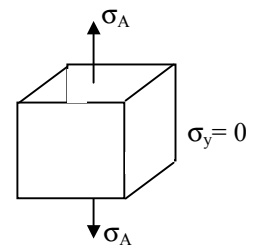
Recall in 2D stress analysis; $\sigma_{1,3} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{zy}^2}$

Point A: $\quad \sigma_{1,3} = \dfrac{156.2 + 0}{2} \pm \sqrt{\left(\dfrac{156.2 - 0}{2}\right)^2 + (35.37)^2}$

$\sigma_1 = 163.84 \ \text{MPa} \qquad\qquad \sigma_2 = 0 \ \text{MPa} \qquad\qquad \sigma_3 = -7.64 \ \text{MPa}$

Point B: $\quad \sigma_{1,3} = \dfrac{-160.7 + 0}{2} \pm \sqrt{\left(\dfrac{-160.7 - 0}{2}\right)^2 + (35.37)^2}$

$\sigma_1 = 7.4 \ \text{MPa} \qquad\qquad \sigma_2 = 0 \ \text{MPa} \qquad\qquad \sigma_3 = -168.1 \ \text{MPa}$
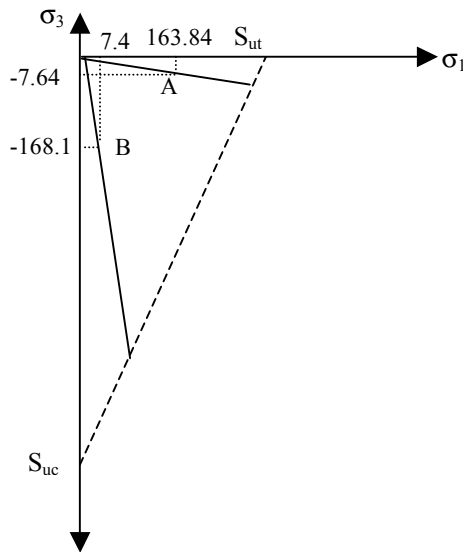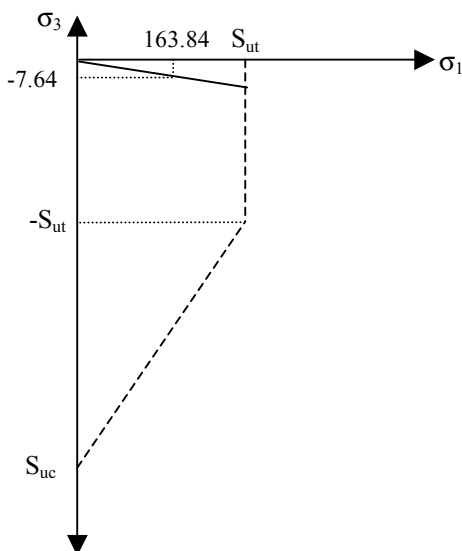
Coulomb-Mohr Theory:

$$\frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} = \frac{1}{n} \quad \text{(note } S_{uc} \text{ is treated as positive)}$$

for point A: $\quad \dfrac{163.84}{325} - \dfrac{-7.64}{912} = \dfrac{1}{n} \qquad \Rightarrow \qquad \underline{n = 1.95} \qquad \text{(minimum)}$
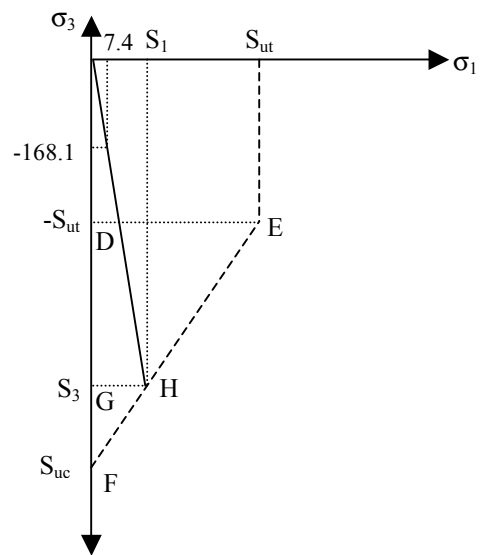
for point B: $\quad \dfrac{7.4}{325} \quad \dfrac{-168.1}{912} = \dfrac{1}{n} \qquad \Rightarrow \qquad n = 4.65$

Modified Mohr Theory:

Point A

Point B

Point A: $n = \dfrac{S_{ut}}{\sigma_1} = \dfrac{325}{163.84} = \underline{1.98}$ (minimum)

Point B: $n = \dfrac{S_1}{\sigma_1} = \dfrac{S_3}{\sigma_3}$ (S_1 and S_3 is to be found)   => $\dfrac{S_1}{7.4} = \dfrac{S_3}{168.1}$

using similarity of triangles DEF and FGH. $\dfrac{S_{ut}}{S_1} = \dfrac{S_{uc} - S_{ut}}{S_{uc} - S_3}$ => $\dfrac{325}{S_1} = \dfrac{912 - 325}{912 - S_3}$

solving the equations : $S_1 = 37.2$ MPa $S_3 = 844.8$ MPa (= -844.8 MPa)

$n = \dfrac{S_1}{\sigma_1} = \dfrac{37.2}{7.4}$ => $n = 5.03$ or $n = \dfrac{S_3}{\sigma_3} = \dfrac{844.8}{168.1}$ $\Rightarrow$ $n = 5.03$

You can also use the equations 12.c, 12.d, 12.e in pp. 274 of Norton to obtain the same result.

Question 4 The steel crankshaft is loaded statically as shown in figure. The steady force is counterbalanced by a twisting torque T and by reactions at A and B. The yield strength of the material is 420 MPa. If the factor of safety according to maximum shear stress theory is to be 2.0, what should be the minimum diameter of the crankshaft? (Note: In practice such problems are dealt with dynamic considerations. Here it is taken as a static example.)

At point C, there is normal stress in axial directiondue to bending (max. moment). At point D, both axial stress due to bending and shear stress due to torsion exist.
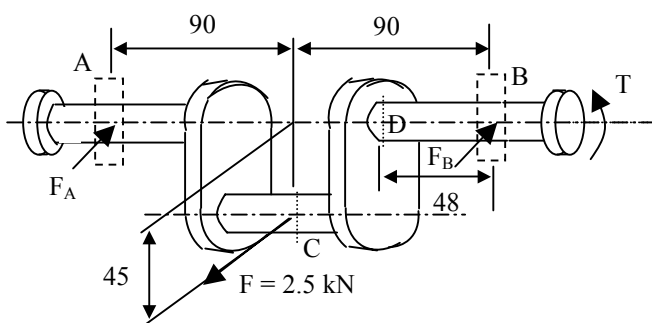
$F_A = F_B = 2500 / 2 = 1250$ N

$T = 2500 \cdot 45 = 1.125 \cdot 10^5$ N.mm

Sections at points C and D should be checked.

At point C:

$M = 1250 \cdot 90 = 1.125 \cdot 10^5$ N.mm $c = \dfrac{d}{2}$ $I = \dfrac{\pi}{64} d^4$

$\sigma_b = \dfrac{Mc}{I} = \dfrac{1.125 \cdot 10^5 \cdot \dfrac{d}{2}}{\dfrac{\pi}{64} d^4} = \dfrac{1.146 \cdot 10^6}{d^3}$ MPa $\sigma_1 = \sigma_b$ $\sigma_2 = \sigma_3 = 0$

Recall , for max shear stress theory: $\tau_{all} = \dfrac{S_y}{2n} = \dfrac{420}{2 \cdot 2} = 105$ MPa   and $\tau_{all} = \tau_{max}$

$\tau_{max} = \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{\dfrac{1.146 \cdot 10^6}{d^3} - 0}{2} = 105$ MPa   => $d^3 = 5457$ mm^3

which gives $d = 17.6$ mm

<u>At point D:</u>

$M = 1250 \cdot 48 = 6 \cdot 10^4$ N.mm $\qquad c = \dfrac{d}{2} \qquad I = \dfrac{\pi}{64} d^4 \qquad J = 2 \cdot I$

$$\sigma_{bending} = \frac{Mc}{I} = \frac{6 \cdot 10^4 \cdot \dfrac{d}{2}}{\dfrac{\pi}{64} d^4} = \frac{6.112 \cdot 10^5}{d^3} \ \ \text{MPa} \qquad\qquad \tau_{tortion} = \frac{Tc}{J} = \frac{1.125 \cdot 10^5 \cdot \dfrac{d}{2}}{\dfrac{\pi}{32} d^4} = \frac{5.73 \cdot 10^5}{d^3}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \tau_{all} = \frac{S_y}{2n} = \frac{420}{2 \cdot 2} = 105 \ \text{MPa} \qquad\qquad \tau_{all} = \tau_{max}$$

$$\tau_{max} = \sqrt{\left(\frac{6.112 \cdot 10^5}{2d^3}\right)^2 + \left(\frac{5.73 \cdot 10^5}{d^3}\right)^2} = \frac{6.494 \cdot 10^5}{d^3} = 105 \ \text{MPa} \qquad => \ d^3 = 6185 \ \text{mm}^3$$

which gives, d = 18.36 mm

Checking both points, point D found to be more critical. The minimum diameter of the shaft should be **18.36 mm**. But it should be better to get used to accept preferred numbers in machine elements design, so it can be set as **d = 20 mm**.