Spatial Error Criterion for Discrete Complex Image Method

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***Abstract*—Although DCIM is very robust and efficient to obtain spatial-domain Green's functions from their spectral-domain counterparts in planar media, the method suffers from the lack of an error metric ensuring the accuracy of the end result, which deters CAD developers and engineers from using it in the development of commercial tools. To remedy this shortcoming of DCIM, a weighted 2-norm of the error in the spatial domain is defined and its mapping to the spectral domain is derived and proposed as the metric to be used in DCIM. In addition, the leakage problem of the multi-level implementations of DCIM is also eliminated. The resulting metric is implemented on the latest three-level DCIM and the proposed three-level DCIM is verified on a typical planar layered geometry.**

I. Introduction

In planar layered media, the spatial-domain Green's functions can be obtained from their closed-form spectral-domain counterparts by evaluating Sommerfeld integral, however their evaluation using numerical integration is computationally quite expensive. DCIM method [1], [2] was developed to obtain the spatial-domain Green's functions in closed forms efficiently and improved several times [3]–[19] and finally became a powerful method with regard to the computational efficiency and the capability of representing all the natural wave constituents of a dipole in any configuration of planar layered media. However all DCIM approaches developed so far suffer from the fact that there is no proper error minimization in the spatial domain. In DCIM, the spectral-domain Green's function is approximated in terms of complex exponentials and the approximation is performed without considering the spatial-domain error, i.e. error-energy minimization in the spectral domain is not equivalent to the error-energy minimization in the spatial domain. Therefore, although many robust and efficient algorithms with good results have been developed, the accuracy of the method has always been questionable [20]. For this reason, engineers avoid using it in the development of commercial softwares and simulation tools. In this study, we propose an algorithm to solve this long-standing shortcoming of the method. For this purpose, a weighted 2-norm of the spatial-domain error is defined and transformed to the spectral domain and a metric is proposed to be used in the implementation of DCIM. It is observed that the Weighted Least Squares (WLS) optimization needs to be employed to achieve the error-energy equivalence of the spatial and spectral domains. The proposed spatial error criterion is applied to the latest three-level DCIM [19] and the spatial-domain Green's functions of vector and scalar potential for a typical planar medium is found by the proposed method for verification.

DCIM with spatial error criterion is proposed and the details of its implementation including its differences from the traditional DCIM are discussed in Section II. Section III provides a typical numerical example to validate the implementation of the algorithm and some conclusions are drawn in Section IV.

II. DCIM and Spatial Error Criterion

The spectral-domain closed-form Green's functions for planar layered media are transformed to the spatial domain by the following Sommerfeld integral, which is also known as Hankel transform integral:

$$g(\rho) = \frac{1}{2\pi} \int_S dk_\rho k_\rho J_0(k_\rho \rho) G(k_\rho), \tag{1}$$

where $g(\rho)$ and $G(k_\rho)$ are the spatial- and spectral-domain Green's functions, respectively, J_0 is a zeroth order Bessel function of the first kind and S denotes the integration path defined in the complex k_ρ-plane. Since the evaluation of Sommerfeld integral by using numerical integration is computationally expensive, DCIM was developed, which is based on *i)* the approximation of the spectral-domain Green's function in terms of complex exponentials

$$G(k_\rho) \cong \hat{G}(k_\rho) = \sum_{l=1}^{M} \alpha^{(l)} \frac{e^{-\beta^{(l)} k_z}}{2jk_z}, \tag{2}$$

where $\hat{G}(k_\rho)$ is the approximation of $G(k_\rho)$, M is the number of exponential terms, $\beta^{(l)}$'s are the constants of the exponents, and $\alpha^{(l)}$'s are the coefficients of the terms, and *ii)* the Sommerfeld integral identity

$$\frac{e^{-jk_0 r}}{r} = \int_S dk_\rho k_\rho J_0(k_\rho \rho) \frac{e^{-jk_z z}}{jk_z}, \tag{3}$$

where $r = \sqrt{\rho^2 + z^2}$ and $k_0 = \sqrt{k_\rho^2 + k_z^2}$. Note that, (2) and (3) are written for a general planar multi-layer structure where the electromagnetic properties vary in z-direction only. By expressing the spectral-domain Green's function as a sum of complex exponentials, the spatial-domain Green's function can be written analytically as a sum of complex images via the Sommerfeld integral identity.

In the standard DCIM algorithm, the computations of $\beta^{(l)}$'s and $\alpha^{(l)}$'s are performed through a subspace approach

(GPOF [21]) and Ordinary Least Squares (OLS), respectively. However, OLS imposes the unweighted energy minimization in the spectral domain which is not equivalent to energy minimization in the spatial domain. This is due to the fact that, unlike Parseval relation for Fourier transform, minimizing unweighted 2-norm in the Hankel transform is not equivalent to minimizing 2-norm in the spatial domain.

A. Spatial Error Criterion

A DCIM based algorithm that minimizes the spatial-domain error needs to achieve a mapping that transforms the error criterion in the spatial domain to the one in the spectral domain. To achieve this goal, we define the spatial-domain error metric as a weighted 2-norm:

$$C(e) = \int_0^\infty \rho|e(\rho)|^2 d\rho, \tag{4}$$

where $e(\rho) = g(\rho) - \hat{g}(\rho)$ is the error function. Here $g(\rho)$ is the true spatial domain function which is unknown and $\hat{g}(\rho)$ is its estimate obtained from DCIM. The weighting function ρ in (4) provides a linear weighting for the contributions of the error values at different locations to the overall error magnitude. Once the cost function is defined in the spatial domain, it needs to be reflected onto the spectral domain involving the error function $E(k_\rho) = G(k_\rho) - \hat{G}(k_\rho)$. The spectral-domain equivalent of the metric in (4) can be obtained by substituting (1) into (4) as

$$\begin{aligned} C(e) =& \int_0^\infty \rho e(\rho) e^*(\rho) d\rho \\ =& \frac{1}{4\pi^2}\int_S\int_S E(k_{\rho 1})E(k_{\rho 2})^* L(k_{\rho 1}, k_{\rho 2}) k_{\rho 1} k_{\rho 2}^* dk_{\rho 1} dk_{\rho 2}^*, \end{aligned} \tag{5}$$

where

$$L(k_{\rho 1}, k_{\rho 2}) = \int_0^{R=\infty} \rho J_0(k_{\rho 1}) J_0^*(k_{\rho 2}) d\rho. \tag{6}$$

The integration kernel L in (6) is the Lommel integral which has a closed-form expression depending on R [22]:

$$L_R(k_{\rho 1}, k_{\rho 2}) = \begin{cases} \frac{1}{k_{\rho 1}^2 - k_{\rho 2}^2}(k_{\rho 2} J_0(k_{\rho 1}R) J_0^*(k_{\rho 2}R) \\ \qquad - k_{\rho 1} J_0(k_{\rho 2}R) J_0^*(k_{\rho 1}R)) & k_{\rho 1} \neq k_{\rho 2}, \\ \frac{k_{\rho 1}^2}{2}(J_0^2(k_{\rho 1}R) + J_1^2(k_{\rho 1}R)) & k_{\rho 1} = k_{\rho 2}. \end{cases} \tag{7}$$

Since the numerical approximation is performed over a discrete set of N k_ρ values, $\mathcal{K}=\{{k_\rho}^{(1)}, {k_\rho}^{(2)}, \ldots, {k_\rho}^{(N)}\}$, in the DCIM approach, (5) can be approximately evaluated for $\mathcal{K}$ as follows:

$$\begin{aligned} \mathcal{C}_{\mathcal{K}}(E) =& \frac{1}{4\pi^2}\mathbf{E}^H \mathbf{\Gamma} \mathbf{E} \\ =& \frac{1}{4\pi^2}\sum\sum_{k_{\rho 1}, k_{\rho 2} \in \mathcal{K}} E(k_{\rho 1}) E^*(k_{\rho 2}) L(k_{\rho 1}, k_{\rho 2}) k_{\rho 1} k_{\rho 2}^* \Delta_{k_{\rho 1}} \Delta_{k_{\rho 2}}^*, \end{aligned} \tag{8}$$

where $\mathbf{E}$ is a vector containing the spectral-domain error values as

$$\mathbf{E} = \begin{bmatrix} E({k_\rho}^{(1)}) & E({k_\rho}^{(2)}) & \ldots & E({k_\rho}^{(N)}) \end{bmatrix}^T, \tag{9}$$

$\mathbf{\Gamma}$ is a square matrix, whose (m,n)'th element is given by

$$\Gamma_{mn} = {k_\rho}^{(n)} {k_\rho}^{(m)*} \Delta_{{k_\rho}^{(n)}} \Delta^*_{{k_\rho}^{(m)}} L({k_\rho}^{(n)}, {k_\rho}^{(m)}), \tag{10}$$

and Δ_{k_ρ} is the distance between consecutive k_ρ samples evaluated forward or backward.

As a result, the spectral-domain mapping of the weighted error metric (referred to as Metric-1) in the spatial domain becomes equivalent to a squared weighted 2-norm,

$$\mathcal{C}_{\mathcal{K}}(E) = \frac{1}{4\pi^2}\|\mathbf{E}\|_{\mathbf{\Gamma}}^2, \tag{11}$$

as $\mathbf{\Gamma}$ is a Hermitian weighting matrix. Note that, for a positive definite matrix $\mathbf{W}$, the weighted 2-norm is defined as

$$\|\mathbf{x}\|_{\mathbf{W}} \triangleq \sqrt{\mathbf{x}^H \mathbf{W} \mathbf{x}}. \tag{12}$$

As a result, incorporation of the spatial-domain error criterion in DCIM algorithm is achieved by employing WLS with the weighting matrix $\mathbf{\Gamma}$ rather than using OLS as in the standard DCIM.

However, especially for large values of R, the arguments of Bessel functions in (7) may grow to an extent that the values of Bessel functions may cause critical inaccuracies. In order to overcome this numerical problem, the weighting matrix $\mathbf{\Gamma}$ can be modified and proposed as Metric-2 as follows: If row-i (or column-i) of $\mathbf{\Gamma}$ requires the evaluation of a Bessel function with argument greater than a chosen threshold τ_{th}, then all the elements in row- and column-i are set to 0 except

$$\Gamma_{ii} = \mathcal{R}e\{{k_\rho}^{(i)}\}\mathcal{R}e\{\Delta_{{k_\rho}^{(i)}}\}. \tag{13}$$

This approximation on the weighting matrix Γ is justified by the orthogonality relation of the Bessel functions [23]. As a more practical approach, the Parseval identity for the Hankel Transform (Metric-3) can be employed:

$$\int_0^\infty \rho|e(\rho)|^2 d\rho = \frac{1}{4\pi^2}\int_0^\infty k|E(k)|^2 dk, \tag{14}$$

which is valid for cases where the integration path S aligns with the real axis in the spectral domain. For the cases where S doesn't align with the real axis, it could be approximated as

$$\int_0^\infty \rho|e(\rho)|^2 d\rho \approx \frac{1}{4\pi^2}\int_S |E(k_\rho)|^2 \mathcal{R}e\{k_\rho\}\mathcal{R}e\{dk_\rho\}, \tag{15}$$

which can be justified by choosing the integration path close to the real axis. This relation also justifies the selection of the weighting function ρ in (4). The spectral-domain weighting matrix corresponding to this case can be written as

$$\begin{aligned} \mathbf{\Gamma} =& diag([\mathcal{R}e\{{k_\rho}^{(1)}\}\mathcal{R}e\{\Delta_{{k_\rho}^{(1)}}\} \quad \mathcal{R}e\{{k_\rho}^{(2)}\}\mathcal{R}e\{\Delta_{{k_\rho}^{(2)}}\} \\ & \ldots \quad \mathcal{R}e\{{k_\rho}^{(N)}\}\mathcal{R}e\{\Delta_{{k_\rho}^{(N)}}\}]). \end{aligned} \tag{16}$$

Note that although there is no spatial selectivity, i.e. R dependency, compared to the weighting matrices of Metric-1 and

Metric-2, it is observed that the approximation in (15) is valid for large R values which is the case in our problem. Therefore, Metric-3 is a convenient choice because of its computational efficiency. Fig. 1 shows the estimates of the spatial-domain weighted energy obtained by the three metric implementations and the true value obtained in the spatial domain, where $g(\rho)$ is found by numerical integration. The weighted energy values in Fig. 1 belong to the Green's function of the vector potential of a typical three-layered planar medium which is a lossy PEC backed material in air with relative permittivity $\epsilon_r = 4-j0.532$ and thickness $d = 10mm$ at 10 GHz [9]. The locations of the observation and the source (HED) points are at the interface between the air and the dielectric layer. It is observed from Fig. 1 that the result of Metric-1 is not reliable for large values of R, as discussed. On the other hand, Metric-2 and Metric-3 solve the numerical problem of Metric-1 and the true value converges to the weighted energy obtained by Metric-3 as R increases. Since we are interested in large R values and Metric-3 is computationally efficient, Metric-3 is chosen to be used in the proposed DCIM throughout this study.

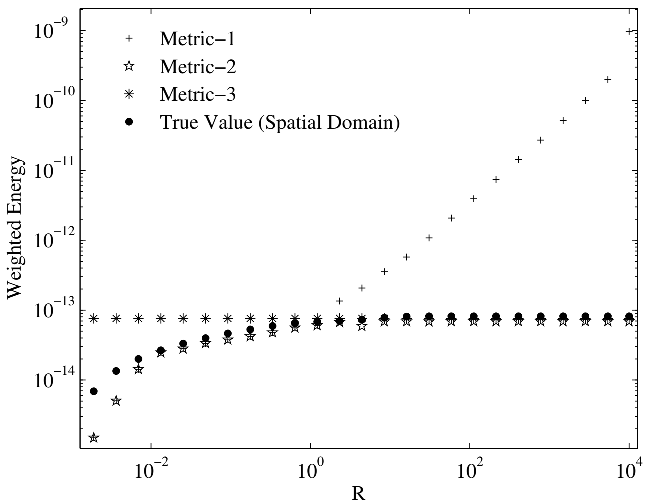

Fig. 1. The spatial-domain weighted energy and its estimates via Metric-1, Metric-2 and Metric-3 as a function of R. To find the true value, the spatial domain Green's function obtained by the numerical integration is taken as $g(\rho)$.

B. Implementation of spatial error criterion on DCIM

In this section, the details of the implementation of the spatial error criterion on DCIM are provided and the leakage issue inherent to the multi-level DCIM approaches is discussed and a simple solution to the leakage is proposed. Recall that the spectral-domain Green's function is approximated in terms of complex exponentials as in (2). In proposed DCIM, $\beta^{(l)}$'s and $\alpha^{(l)}$'s are found by employing spatial error criterion according to the optimization

$$\text{minimize} \quad \|\mathbf{G} - \hat{\mathbf{G}}\|^2_{\mathbf{\Gamma}}. \tag{17}$$

Note that, this nonlinear least squares problem is solved by DCIM in two steps. First, GPOF is applied to $H(k_\rho) = jk_zG(k_\rho)$ to find $\beta^{(l)}$'s and then the linear least squares is applied to find $\alpha^{(l)}$'s. Since GPOF procedure is a non-linear approximation, the spatial error criterion is employed to the second part of DCIM. To clarify the differences between the traditional and proposed DCIM approaches, their main steps are listed below:

- Standard DCIM: $\beta^{(l)}$'s are found from GPOF and $\alpha^{(l)}$'s are computed by solving OLS problem:

$$\text{minimize} \quad \|\mathbf{H} - \hat{\mathbf{H}}\|^2, \tag{18}$$

where

$$\mathbf{H} = \left[H({k_\rho}^{(1)}) \;\; H({k_\rho}^{(2)}) \;\; \ldots \;\; H({k_\rho}^{(N)})\right]^T, \tag{19}$$

$$H({k_\rho}^{(n)}) = 2j{k_z}^{(n)}G({k_\rho}^{(n)}), \qquad n = 1, \ldots, N \tag{20}$$

and

$$\hat{\mathbf{H}} = \left[\hat{H}({k_\rho}^{(1)}) \;\; \hat{H}({k_\rho}^{(2)}) \;\; \ldots \;\; \hat{H}({k_\rho}^{(N)})\right]^T, \tag{21}$$

$$\begin{aligned}\hat{H}({k_\rho}^{(n)}) &= 2j{k_z}^{(n)}\hat{G}({k_\rho}^{(n)}) \\ &= \sum_{l=1}^{M} \alpha^{(l)} e^{-\beta^{(l)}{k_z}^{(n)}}, \quad n = 1, \ldots N.\end{aligned} \tag{22}$$

Therefore, $\alpha^{(l)}$'s are obtained by matching the jk_z-scaled version of the spectrum via the solution of the OLS problem as

$$\alpha_{OLS} = \mathbf{A}^{\dagger}\mathbf{H}, \tag{23}$$

where $\mathbf{A}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$, which is $N \times M$ matrix with (i,j)'th element

$$\mathbf{A}_{ij} = e^{-\beta^{(j)}{k_z}^{(i)}}. \tag{24}$$

- Proposed DCIM: $\beta^{(l)}$'s are found from GPOF and $\alpha^{(l)}$'s are computed by solving WLS problem:

$$\text{minimize} \quad \|\mathbf{G} - \hat{\mathbf{G}}\|^2_{\mathbf{\Gamma}}, \tag{25}$$

and its solution can be compactly written as

$$\alpha_{WLS} = (\mathbf{\Gamma}^{1/2}\mathbf{B})^{\dagger}(\mathbf{\Gamma}^{1/2}\mathbf{G}), \tag{26}$$

where $\mathbf{B}$ is $N \times M$ matrix with (i,j)'th element

$$\mathbf{B}_{ij} = \frac{e^{-\beta^{(j)}{k_z}^{(i)}}}{2j{k_z}^{(i)}} \tag{27}$$

and ${\mathbf{\Gamma}^{1/2}}^H\mathbf{\Gamma}^{1/2} = \mathbf{\Gamma}$.

Consequently, there are two major differences between the standard DCIM and the proposed DCIM. First, WLS optimization is employed instead of OLS and second, fitting is applied to the spectral domain Green's function rather than its scaled function.

As an additional improvement on DCIM, the leakage problem which occurs in multi-level implementations of DCIM is eliminated. In a typical multi-level DCIM, the sampling domain is divided into sub-regions and DCIM is applied in each sub-region. Regions are processed in order, starting from the sub-region with highest end of the spectrum to the one with lowest end in a semi-independent way. Therefore the leakage from low- to high-frequency levels can occur and the leakage can even exceed the fitting error, which may mislead the user about the overall error of the fitting. To eliminate this problem, the whole DCIM algorithm is implemented as usual, except for the final WLS optimization that provides the weight parameters ($\alpha^{(l)}$) for the last region. At this stage, all $\alpha^{(l)}$'s computed in the previous steps are completely ignored, and a new WLS optimization is applied over all samples that were already used, to obtain the weight parameters for all the exponentials found by the multilevel procedure.

III. Results and Discussions

In this section, the three-layered medium used in the previous section is studied for the sake of coherence and its spatial-domain Green's functions of vector and scalar potential found by three-level DCIM with spatial error criterion are demonstrated. For validation of its performance, the results found by numerical integration and the standard DCIM are also presented. It's seen that the proposed DCIM predicts the nature of the Green's function better than the standard DCIM does, since the proposed DCIM assures the error minimization in the spatial domain. The geometry has a single TE-mode and two TM-mode surface wave (SW) poles which are complex due to the lossy medium. Since their contribution decays exponentially in far field, but they still make the fitting slightly more difficult, the DCIM approaches are employed without extracting them to compare the performance of the methods more apparently. Fig. 2 and Fig. 3 demonstrate the spatial-domain Green's function of vector potential obtained by the standard DCIM and the proposed DCIM, respectively. In addition, the contribution of the SW poles is also demonstrated in figures. As seen, the proposed DCIM is quite successful and coincides perfectly with the result of the numerical integration, on the other hand, the standard DCIM fails in fitting. To demonstrate the performance of the method for the spatial-domain Green's function of scalar potential, Fig. 4 and Fig. 5 are presented for standard and proposed DCIM, respectively. Similarly, while the standard DCIM fails towards far field, the proposed DCIM finds the Green's function correctly. It should be noted that, the results found by the standard DCIM can be improved when SW poles are extracted, yet the standard DCIM does not assure the error minimization in the spatial domain and it still tends to fail in far field. This examples also show that even tough the SW poles (at least the complex ones) are not subtracted before the application of DCIM and therefore the behavior of the spectral-domain Green's function is relatively more complicated to be approximated, the proposed DCIM is still capable of finding the spatial-domain Green's function correctly. This is due to the fact that, together with leakage elimination, the contribution of the error values towards far field is enhanced by using weighted least squares and consequently the fitting is improved for those regions.

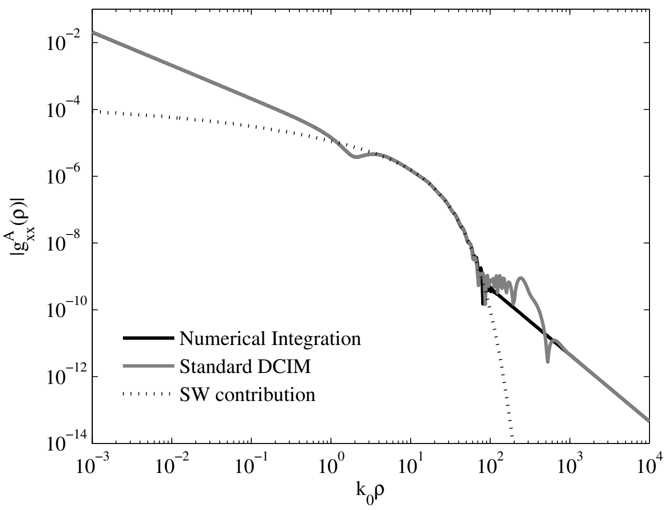

Fig. 2. Magnitude of the spatial-domain Green's function for the vector potential obtained by the standard DCIM. Geometry: A lossy PEC backed material in air with relative permittivity $\epsilon_r = 4 - j0.532$ and thickness $d = 10mm$ at 10 GHz.

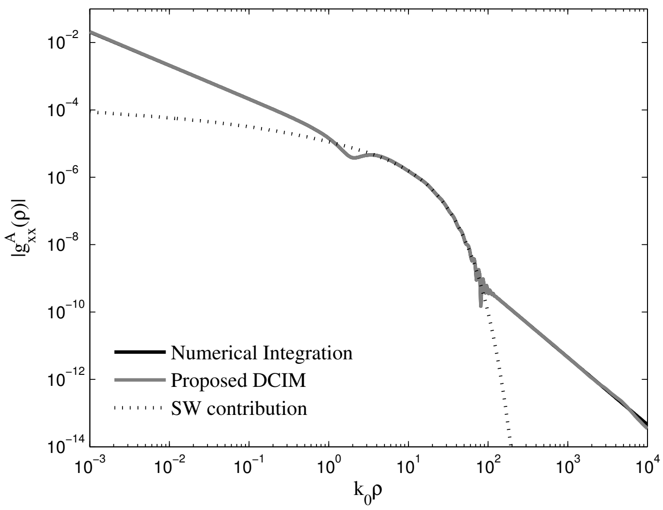

Fig. 3. Magnitude of the spatial-domain Green's function for the vector potential obtained by the proposed DCIM. Geometry: A lossy PEC backed material in air with relative permittivity $\epsilon_r = 4 - j0.532$ and thickness $d = 10mm$ at 10 GHz.

IV. Conclusion

The lack of an error metric that transforms the spatial-domain error to the spectral domain is an important shortcoming of the otherwise very efficient and hence very popular DCIM approach and it is remedied in this study. In standard DCIM procedures, OLS error minimization in the spectral domain does not correspond to the error minimization in the spatial domain. We propose an error criterion by defining an error metric in the spatial domain and mapping it to the spectral domain. In addition to introducing an error criterion, a simple solution to the leakage problem of the

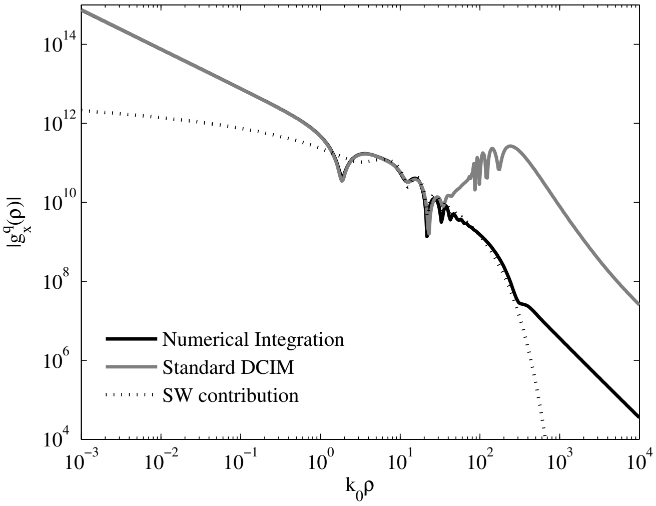

Fig. 4. Magnitude of the spatial-domain Green's function of the scalar potential obtained by the standard DCIM. Geometry: A lossy PEC backed material in air with relative permittivity $\epsilon_r = 4 - j0.532$ and thickness $d = 10mm$ at 10 GHz.

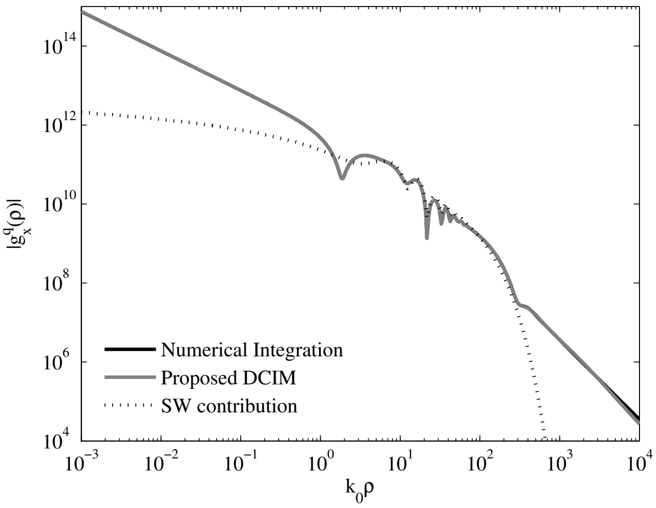

Fig. 5. Magnitude of the spatial-domain Green's function of the scalar potential obtained by the proposed DCIM. Geometry: A lossy PEC backed material in air with relative permittivity $\epsilon_r = 4 - j0.532$ and thickness $d = 10mm$ at 10 GHz.

semi-independent processing of sub-regions in multi-level implementations of DCIM is proposed. The resulting DCIM algorithm is implemented on three-level DCIM and tested on a typical geometry for validation.

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