Spectral Singularities, Unidirectional Invisibility, and Dynamical Formulation of 1-Dim. Scattering Theory

Ali Mostafazadeh

Koç University, Istanbul

Outline:

-Motivation: **Pseudo-Hermitian QM**

-**Transfer Matrix**, **Spectral Singularities**, & **Unidirectional Invisibility**

- **Transfer Matrix** as a **Non-Unitary S-Matrix**

- **Dynamical Equation for Transfer Matrix**

- **Adiabatic Approximations, Semiclassical Scattering** & **Geometric Phases**

- **Local Inverse Scattering**

Motivation:

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

Motivation:

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

- This operator commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) := \psi(-x), \quad \mathcal{T}\psi(x) := \psi(x)^*$$

Motivation:

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

- This operator commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) := \psi(-x), \qquad \mathcal{T}\psi(x) := \psi(x)^*$$

- $H = ix$ is $\mathcal{PT}$-symmetric but its spectrum is imaginary.

- What about non-$\mathcal{PT}$-symmetric operators such as

$$H = (p + \mathfrak{z}\, x)^2 + x^2 \quad \text{or} \quad H = p^2 + \mathfrak{z}\, \delta(x) \quad \text{with } \mathfrak{z} \in \mathbb{C},$$

which have a real spectrum?

Motivation:

Bender & Boettcher, PRL **80**, 5243 (1998):

$H = p^2 + ix^3$ has a real spectrum. Why?

- This operator commutes with $\mathcal{PT}$, where

$$\mathcal{P}\psi(x) := \psi(-x), \quad \mathcal{T}\psi(x) := \psi(x)^*$$

- $H = ix$ is $\mathcal{PT}$-symmetric but its spectrum is imaginary.

- What about non-$\mathcal{PT}$-symmetric operators such as

$$H = (p + \mathfrak{z}\, x)^2 + x^2 \quad \text{or} \quad H = p^2 + \mathfrak{z}\, \delta(x) \quad \text{with } \mathfrak{z} \in \mathbb{C},$$

which have a real spectrum?

- Can we use such operators in QM as Hamiltonians operators or observables?

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H \eta_+{}^{-1}$ for some positive-definite metric operator η_+.

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H {\eta_+}^{-1}$ for some positive-definite metric operator η_+.

In this case it becomes Hermitian if we change the inner product of the Hilbert space:

$$\langle\psi|\phi\rangle \;\to\; \langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle$$

Measurement Axiom of QM $\Rightarrow$ Observables are diagonalizable operators with real spectrum.

Let $H : \mathscr{H} \to \mathscr{H}$ be an operator with a discrete spectrum.

Diagonalizability of H means the existence of a complete biorthonormal eigensystem $\{(\phi_n, \psi_n)\}$:

$$H\psi_n = E_n\psi_n, \quad H^\dagger\phi_n = E_n^*\phi_n, \quad \langle\phi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_n |\psi_n\rangle\langle\phi_n| = 1$$

Qxn: What are the necessary and sufficient conditions for an operator to be diagonalizable and have a real spectrum.

Ans: It must satisfy $H^\dagger = \eta_+ H \eta_+^{-1}$ for some positive-definite metric operator η_+.

In this case it becomes Hermitian if we change the inner product of the Hilbert space:

$$\langle\psi|\phi\rangle \;\to\; \langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle \qquad \eta_+ = \sum_n |\phi_n\rangle\langle\phi_n|$$

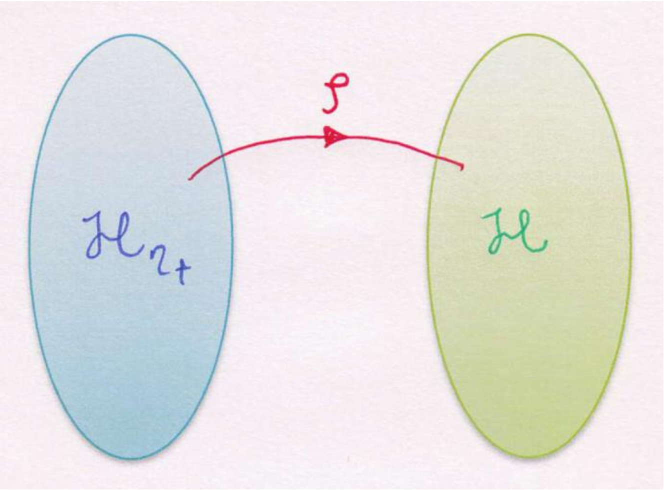
A.M., "Pseudo-Hermiticity versus $\mathcal{PT}$-Symmetry I, II, III," JMP **43**, 205, 2814, 3944 (2002).

Pseudo-Hermitian QM:

- Use $\langle\cdot,\cdot\rangle_{\eta_+}$ to construct a Hilbert space, $\mathscr{H}_{\eta_+}$.
- Because $H:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$ is Hermitian, $(\mathscr{H}_{\eta_+},H)$ defines a unitary quantum system.

Pseudo-Hermitian QM:

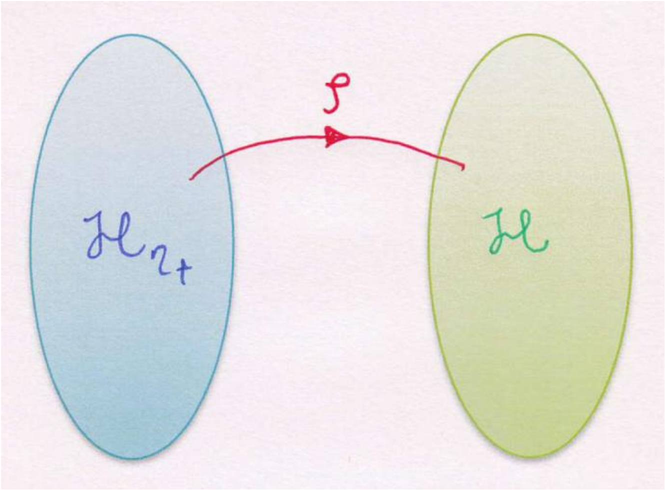
- Use $\langle\cdot,\cdot\rangle_{\eta_+}$ to construct a Hilbert space, $\mathscr{H}_{\eta_+}$.
- Because $H:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$ is Hermitian, $(\mathscr{H}_{\eta_+},H)$ defines a unitary quantum system.
- $\rho:=\sqrt{\eta_+}:\mathscr{H}_{\eta_+}\to\mathscr{H}$ is a unitary operator.
- $h:=\rho\,H\,\rho^{-1}:\mathscr{H}\to\mathscr{H}$ is Hermitian.



$(\mathscr{H}_{\eta_+},H)$ and $(\mathscr{H},h)$ are unitary-equivalent.

Pseudo-Hermitian QM:

- Use $\langle\cdot,\cdot\rangle_{\eta_+}$ to construct a Hilbert space, $\mathscr{H}_{\eta_+}$.
- Because $H : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$ is Hermitian, $(\mathscr{H}_{\eta_+}, H)$ defines a unitary quantum system.
- $\rho := \sqrt{\eta_+} : \mathscr{H}_{\eta_+} \to \mathscr{H}$ is a unitary operator.
- $h := \rho\, H\, \rho^{-1} : \mathscr{H} \to \mathscr{H}$ is Hermitian.



$(\mathscr{H}_{\eta_+}, H)$ and $(\mathscr{H}, h)$ are unitary-equivalent.

- The basic ingredient is the metric operator η_+.

A. M. Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010); **arXiv:0810.5643**.

Examples:

$$H = \frac{p^2}{2m} + \frac{\mu^2}{2}x^2 + \mathrm{i}\epsilon x^3$$

$$h = \frac{p^2}{2m} + \frac{1}{2}\mu^2 x^2 + \frac{3}{2\mu^4}\left(\frac{1}{m}\{x^2, p^2\} + \mu^2 x^4 + \frac{2\hbar^2}{3m}\right)\epsilon^2 + \frac{2}{\mu^{12}}\left(\frac{p^6}{m^3} - \frac{9\mu^2}{m^2}\{x^2, p^4\}\right.$$
$$\left. - \frac{51\mu^4}{8m}\{x^4, p^2\} - \frac{7\mu^6}{4}x^6 - \frac{81\hbar^2\mu^2}{2m^2}p^2 - \frac{69\hbar^2\mu^4}{2m}x^2\right)\epsilon^4 + \mathcal{O}(\epsilon^6)$$

[JPA 38 (2005) 6557 & 39 (2006) 13495]

δ-Function Potential with Complex Coupling

$$H = \frac{p^2}{2m} + z\,\delta(x), \qquad \Re(z) > 0$$

H is not $\mathcal{PT}$-symmetric.

δ-Function Potential with Complex Coupling

$$H = \frac{p^2}{2m} + z\,\delta(x), \qquad \Re(z) > 0$$

H is not $\mathcal{PT}$-symmetric.

$$h = \frac{p^2}{2m} + \Re(z)\delta(x) + h_2\,\Im(z)^2 + \mathcal{O}(\Im(z))^3$$

$$h_2\Psi(x) := A_\psi\, e^{-|x|/L} + B_\psi\,\delta(x)$$

$$A_\psi := \frac{m\Psi(0)}{8\hbar^2}, \qquad B_\psi = \frac{m}{8\hbar^2}\int_{-\infty}^{\infty} dx\; e^{-|x|/L}\Psi(x).$$

$$L := \frac{\hbar^2}{m\Re(z)}$$

[A. M., JPA 39 (2006) 13506]

δ-Function Potential with Complex Coupling

$$H = \frac{p^2}{2m} + z\,\delta(x), \qquad \Re(z) > 0$$

H is not $\mathcal{PT}$-symmetric.

$$h = \frac{p^2}{2m} + \Re(z)\delta(x) + h_2\,\Im(z)^2 + \mathcal{O}(\Im(z))^3$$

$$h_2\Psi(x) := A_\psi\, e^{-|x|/L} + B_\psi\,\delta(x)$$

$$A_\psi := \frac{m\Psi(0)}{8\hbar^2}, \qquad B_\psi = \frac{m}{8\hbar^2}\int_{-\infty}^{\infty} dx\; e^{-|x|/L}\Psi(x).$$

$$L := \frac{\hbar^2}{m\Re(z)}$$

What happens if $\Re(z) = 0$?

1-Dim. Scattering & Spectral Singularities

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent potential &

$$v(x) \to 0 \quad \text{for} \quad x \to \pm\infty.$$

1-Dim. Scattering & Spectral Singularities

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent potential &

$$v(x) \to 0 \quad \text{for} \quad x \to \pm\infty.$$

- Asymptotic solutions:

$$\psi(x) = A_\pm e^{ikx} + B_\pm e^{-ikx} \quad \text{for} \quad x \to \pm\infty.$$

- Transfer matrix: $\left[\begin{array}{c} A_+ \\ B_+ \end{array}\right] = \left[\begin{array}{cc} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{array}\right] \left[\begin{array}{c} A_- \\ B_- \end{array}\right].$
- $\det \mathbf{M} = 1.$

Spectral Singularities are the real zeros of $M_{22}(k)$.

- Example: $v(x) = z\,\delta(x)$, $z \in \mathbb{C}$:

  - Transfer matrix: $\mathbf{M} = \begin{bmatrix} 1 - \frac{iz}{2k} & -\frac{iz}{2k} \\ \frac{iz}{2k} & 1 + \frac{iz}{2k} \end{bmatrix}$

  - There is a spectral singularity for $z \in i\mathbb{R}$ at $k^2 = -\dfrac{z^2}{4}$.

[A. M., JPA 39 (2006) 13506]

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$T^l = T^r =: T$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$T^l = T^r =: T$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Reflection and Transmission amplitudes diverge at a spectral singularity.

$\Rightarrow$ infinite amplification of incident waves.

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

$$T^l = T^r =: T$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}$$

- Reflection and Transmission amplitudes diverge at a spectral singularity.

$\Rightarrow$ infinite amplification of incident waves.

- Physically they correspond to scattering states that behave like resonances: Zero-width resonances.

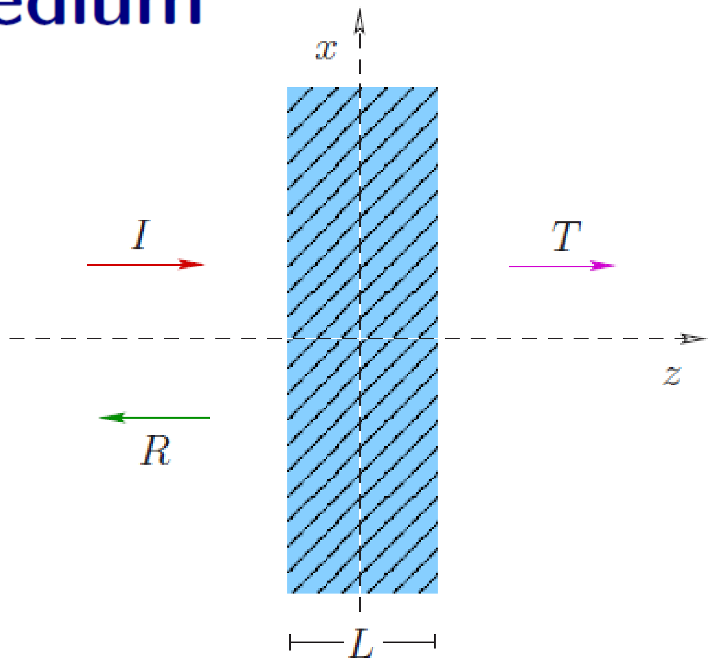
[PRL **102**, 220402 (2009); arXiv:0901.4472]

A Physical Application:

Infinite planar slab gain medium

$$\mathfrak{n}^2\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index

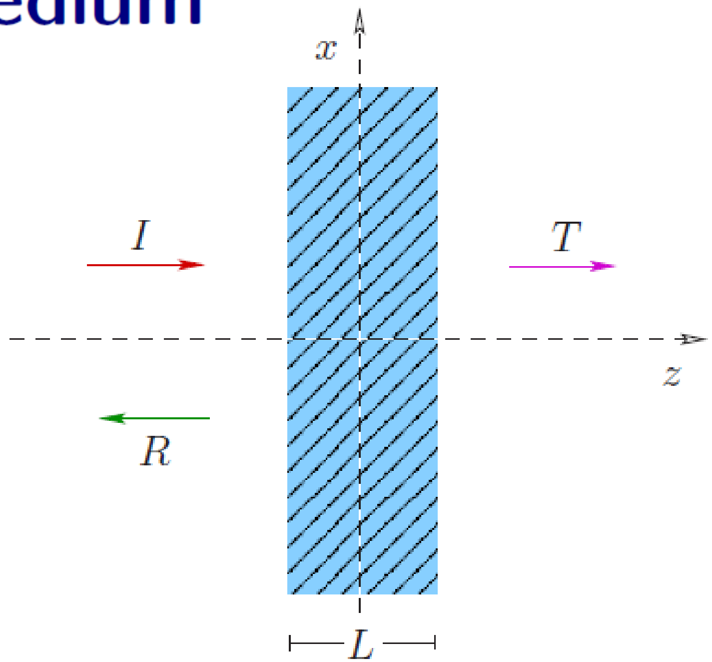


A Physical Application:

Infinite planar slab gain medium

$$\mathfrak{n}^2 \partial_t^2 \vec{E} - c^2 \partial_z^2 \vec{E} = 0$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index



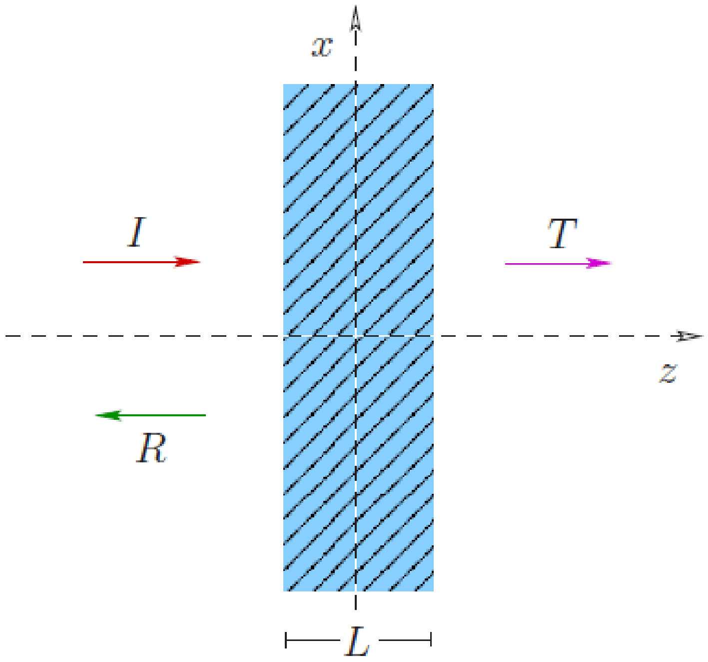
$$\vec{E} \propto e^{\frac{i\omega}{c}(\mathfrak{n}z - ct)}\,\hat{\imath} = e^{-\frac{\omega\kappa z}{c}}\, e^{\frac{i\omega}{c}(\eta z - ct)}\,\hat{\imath}$$

$|\vec{E}|^2$ grows exponentially for $\kappa < 0$: Gain Medium

Gain Coefficient: $g := -\frac{2\omega\kappa}{c}$

$$\mathfrak{n}^2 \partial_t^2 \vec{E} - c^2 \partial_z^2 \vec{E} = 0$$

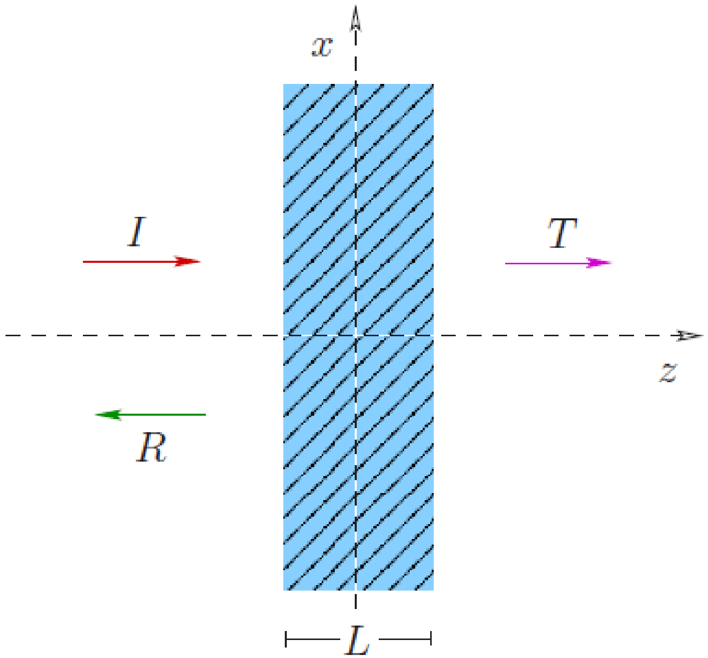
$$\vec{E}(z,t) = E\, e^{-i\omega t}\, \psi(z)\, \hat{\imath}$$



$$\mathfrak{n}^2\partial_t^2\vec{E} - c^2\partial_z^2\vec{E} = 0$$

$$\vec{E}(z,t) = E\, e^{-i\omega t}\,\psi(z)\,\hat{\imath}$$

$$-\psi''(z) + v(z)\psi(z) = k^2\psi(z)$$



Complex Barrier Potential:

$$v(z) := \begin{cases} \mathfrak{z} & \text{for} \quad |z| \leq L/2 \\ 0 & \text{for} \quad |z| > L/2 \end{cases}$$

$$\mathfrak{z} := k^2(1-\mathfrak{n}^2) \in \mathbb{C}, \qquad k := \omega/c$$

$\mathfrak{n} := \eta + i\kappa$: Refractive index

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Optical Spectral Singularity $\Rightarrow$ Lasing at threshold gain

[A. M. PRA **83**, 045801 (2011); arXiv:1102.4695]

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L} \ln \frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Optical Spectral Singularity $\Rightarrow$ **Lasing at threshold gain**

[A. M. PRA **83**, 045801 (2011); arXiv:1102.4695]

Time-reversal transformation: $\mathbf{M} \xrightarrow{\mathcal{T}} \sigma_1 \mathbf{M}^* \sigma_1$

$$\Rightarrow M_{22} \xleftrightarrow{\mathcal{T}} M_{11}^*.$$

Spectral singularities of complex barrier potential:

$$M_{22} = 0 \quad \Rightarrow \quad e^{-2i\mathfrak{n}Lk} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$$

$\mathcal{R} := \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2$ is called Reflectivity in optics.

Taking Absolute value of both sides gives:

$$e^{-2gL} = |\mathcal{R}|^2 \quad \Rightarrow \quad g = \frac{1}{2L}\ln\frac{1}{|\mathcal{R}|^2}$$

This is known as the Laser Threshold Condition.

Optical Spectral Singularity $\Rightarrow$ **Lasing at threshold gain**

[A. M. PRA **83**, 045801 (2011); arXiv:1102.4695]

Time-reversal transformation: $\mathbf{M} \xrightarrow{\mathcal{T}} \sigma_1 \mathbf{M}^* \sigma_1$

$$\Rightarrow M_{22} \xleftrightarrow{\mathcal{T}} M_{11}^*.$$

- Time-reversed optical spectral singularities:

$\Rightarrow$ Antilasing (CPA) [Wan et al Science 2010]

Unidirectional Invisibility

$$\mathbf{M} = \begin{bmatrix} T - \frac{R^l R^r}{T} & \frac{R^r}{T} \\ -\frac{R^l}{T} & \frac{1}{T} \end{bmatrix}$$

Unidir. Reflectionlessness: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$

Only one of M_{12} and M_{21} is zero.

Unidir. Invisibility: $R^l = 0 \neq R^r$ or $R^r = 0 \neq R^l$ & $T = 1$

Only one of M_{12} and M_{21} is zero & $M_{11} = M_{22} = 1$.

Lin et al, PRL **106**, 213901 (2011)
Regensburger et al, Nature **488**, 167 (2012)
A. M. PRA **87**, 012103 (2013)

If $v(x)$ is a real potential,

$$|R^r| = |R^l|, \qquad |R^{l/r}|^2 + |T|^2 = 1$$

$\Rightarrow$ Spectral singularities and unidirectional reflectionlessness & invisibility cannot happen for a real potential.

Composition Property of M

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$

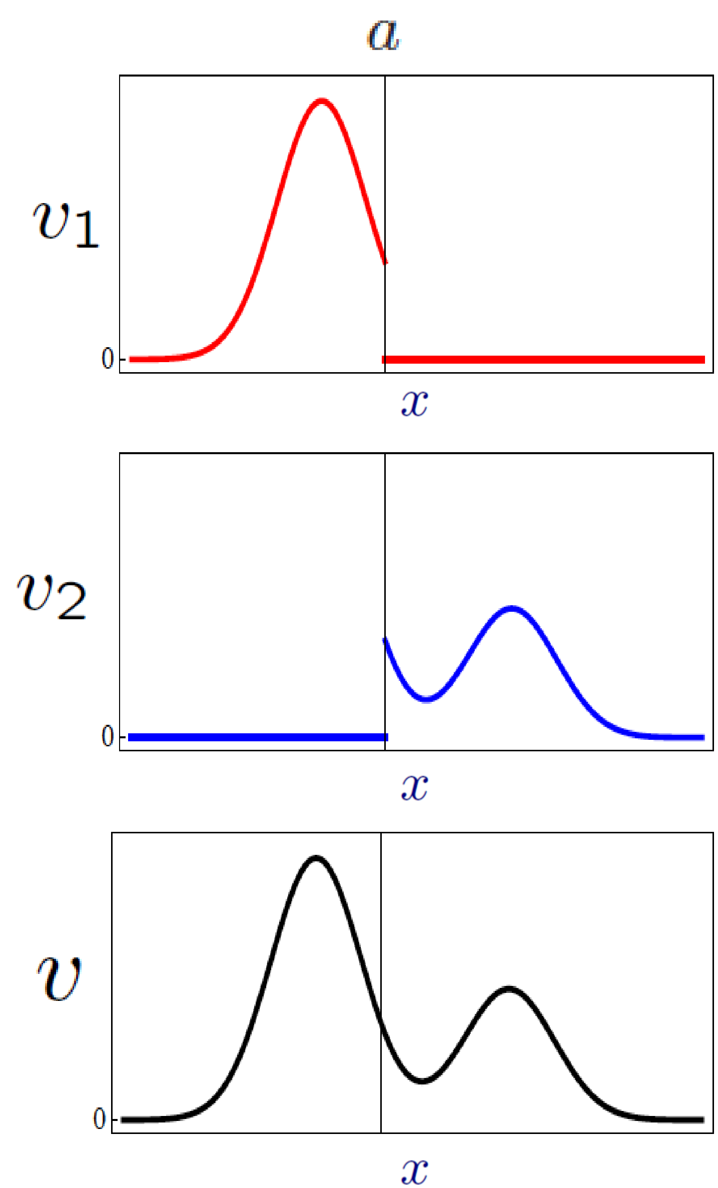
$$v_2(x) = 0 \quad \text{for} \quad x < a$$

$$v(x) = v_1(x) + v_2(x).$$

M_1: Transfer matrix of v_1

M_2: Transfer matrix of v_2

M: Transfer matrix of $v = v_1 + v_2$



Composition Property of $\mathbf{M}$

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$

$$v_2(x) = 0 \quad \text{for} \quad x < a$$

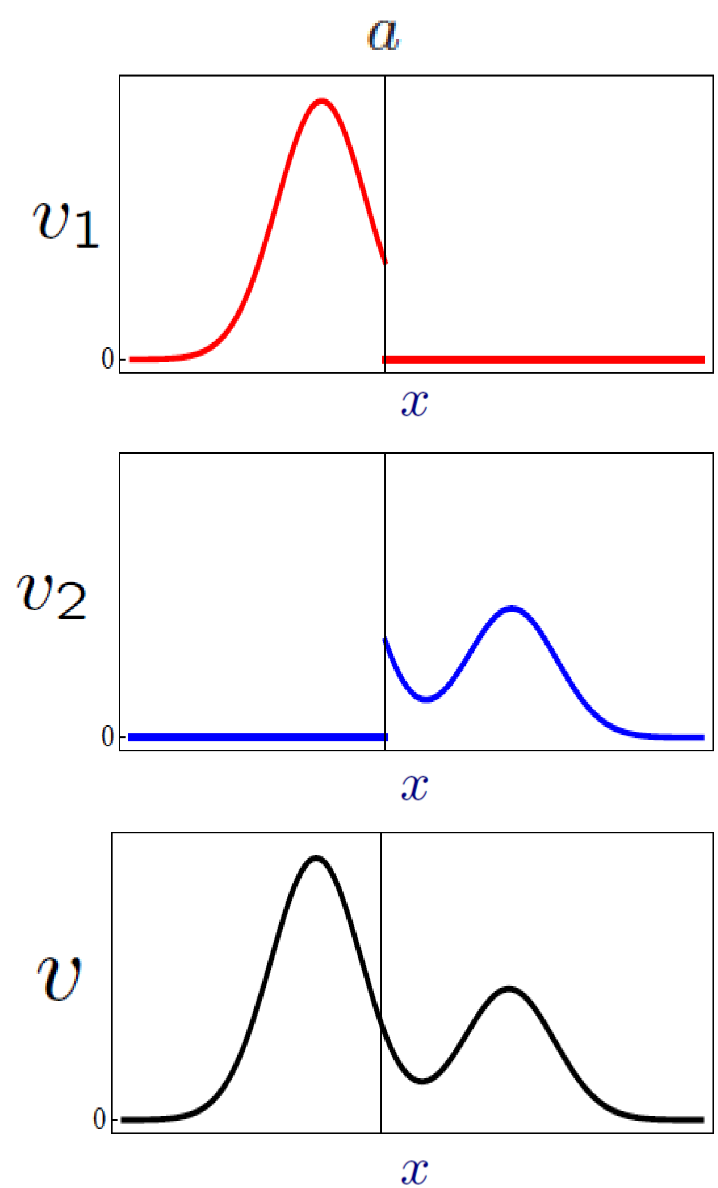
$$v(x) = v_1(x) + v_2(x).$$

$\mathbf{M}_1$: Transfer matrix of v_1

$\mathbf{M}_2$: Transfer matrix of v_2

$\mathbf{M}$: Transfer matrix of $v = v_1 + v_2$

Then $\mathbf{M} = \mathbf{M}_2 \mathbf{M}_1$.



Composition Property of $\mathbf{M}$

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$

$$v_2(x) = 0 \quad \text{for} \quad x < a$$

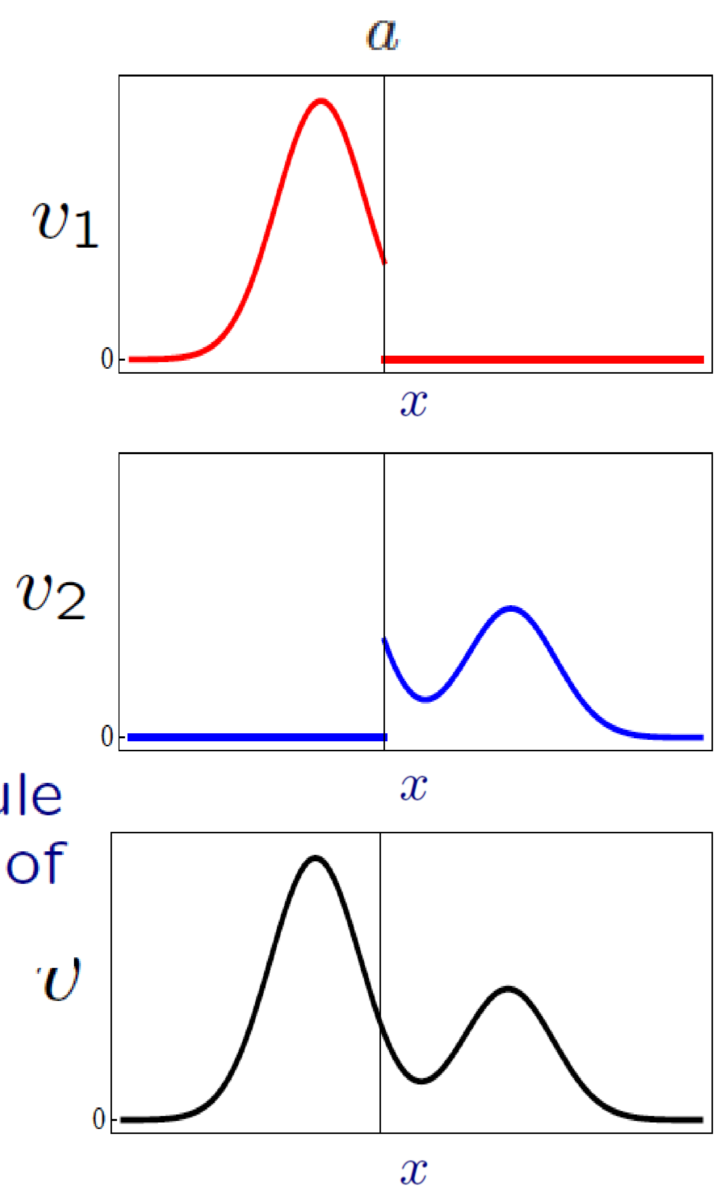
$$v(x) = v_1(x) + v_2(x).$$

$\mathbf{M}_1$: Transfer matrix of v_1

$\mathbf{M}_2$: Transfer matrix of v_2

$\mathbf{M}$: Transfer matrix of $v = v_1 + v_2$

Then $\mathbf{M} = \mathbf{M}_2 \mathbf{M}_1$.



This resembles the composition rule for the evolution operator $U(t, t_0)$ of a quantum system:

$$U(t_2, t_0) = U(t_2, t_1) U(t_1, t_0)$$

for $t_0 \leq t_1 \leq t_2$.

Time-indep. Schrödinger Eq.: $-\psi''(x) + v(x)\psi(x) = k^2\psi(x)$

$$i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(\tau)$$

Time-indep. Schrödinger Eq.: $-\psi''(x) + v(x)\psi(x) = k^2\psi(x)$

$$\tau := kx, \qquad \phi(\tau) := \psi(\tau/k), \qquad \dot{\phi}(\tau) := \frac{d\phi(\tau)}{d\tau}, \qquad w(\tau) := \frac{v(\frac{\tau}{k})}{2k^2}$$

$$\Psi(\tau) := \frac{1}{2}\begin{bmatrix} \phi(\tau) - i\dot{\phi}(\tau) \\ \phi(\tau) + i\dot{\phi}(\tau) \end{bmatrix}, \quad \mathbf{N} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = i\sigma_2 + \sigma_3$$

$$\mathbf{H}(\tau) := \begin{bmatrix} w(\tau) - 1 & w(\tau) \\ -w(\tau) & -w(\tau) + 1 \end{bmatrix} = -\sigma_3 + w(\tau)\mathbf{N}$$

$$i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(\tau)$$

Time-indep. Schrödinger Eq.: $-\psi''(x) + v(x)\psi(x) = k^2\psi(x)$

$$\mathbf{H}(\tau) \;:=\; \begin{bmatrix} w(\tau)-1 & w(\tau) \\ -w(\tau) & -w(\tau)+1 \end{bmatrix}$$

$$i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(\tau)$$

- $\mathbf{H}(\tau)$ is a 2-level non-Hermitian Hamiltonian.
- $\mathbf{H}(\tau)$ is σ_3-pseudo-Hermitian, if v is real; $\mathbf{H}(\tau)^\dagger = \sigma_3\mathbf{H}(\tau)\sigma_3^{-1}$.
- Eigenvalues of $\mathbf{H}(\tau) = \pm\mathfrak{n}(\tau)$, $\mathfrak{n} := \sqrt{1-2w} = \sqrt{1-v/k^2}$.
- Classical turning points are exceptional points of $\mathbf{H}(\tau)$.

$$\mathbf{H}(\tau) := -\sigma_3 + w(\tau)\mathbf{N}, \qquad \mathbf{N} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

- Evolution operator: $\mathbf{U}(\tau, \tau_0) := \mathscr{T} e^{-i\int_{\tau_0}^{\tau} \mathbf{H}(t)dt}$;

$$i\dot{\mathbf{U}}(\tau, \tau_0) = \mathbf{H}(\tau)\mathbf{U}(\tau, \tau_0), \qquad \mathbf{U}(\tau_0, \tau_0) = 1$$

$$\Psi(\tau) = \mathbf{U}(\tau, \tau_0)\Psi(\tau_0)$$

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathbf{U}(\tau, \tau_0) = \mathbf{U}_0(\tau - \tau_0)$.

$$\mathbf{U}_0(\tau) := e^{i\tau\sigma_3}$$

$$\mathbf{H}(\tau) := -\sigma_3 + w(\tau)\mathbf{N}, \qquad \mathbf{N} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

- Evolution operator: $\mathbf{U}(\tau, \tau_0) := \mathscr{T} e^{-i\int_{\tau_0}^{\tau} \mathbf{H}(t)dt}$;

$$i\dot{\mathbf{U}}(\tau, \tau_0) = \mathbf{H}(\tau)\mathbf{U}(\tau, \tau_0), \qquad \mathbf{U}(\tau_0, \tau_0) = 1$$

$$\Psi(\tau) = \mathbf{U}(\tau, \tau_0)\Psi(\tau_0)$$

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathbf{U}(\tau, \tau_0) = \mathbf{U}_0(\tau - \tau_0)$.

$$\mathbf{U}_0(\tau) := e^{i\tau\sigma_3}$$

Theorem: The S-matrix of $\mathbf{H}(\tau)$ is the transfer matrix of v;

$$\mathbf{M} = \mathbf{U}_0(+\infty)^{-1}\mathbf{U}(+\infty, -\infty)\mathbf{U}_0(-\infty).$$

A. M. PRA. **89**, 012709 (2014)

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathbf{U}(\tau, \tau_0) = \mathbf{U}_0(\tau - \tau_0)$.
- Interaction-picture Hamiltonian: $\Psi(\tau) \to U_0(\tau)^{-1}\Psi(\tau)$

$$
\begin{aligned}
\mathscr{H}(\tau) &:= \mathbf{U}_0(\tau)^{-1}\mathbf{H}(\tau)\mathbf{U}_0(\tau) - i\mathbf{U}_0(\tau)^{-1}\dot{\mathbf{U}}_0(\tau) \\
&= w(\tau)\begin{bmatrix} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{bmatrix}
\end{aligned}
$$

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathrm{U}(\tau, \tau_0) = \mathrm{U}_0(\tau - \tau_0)$.
- Interaction-picture Hamiltonian: $\Psi(\tau) \to U_0(\tau)^{-1}\Psi(\tau)$

$$\begin{aligned} \mathscr{H}(\tau) &:= \mathrm{U}_0(\tau)^{-1}\mathbf{H}(\tau)\mathrm{U}_0(\tau) - i\mathrm{U}_0(\tau)^{-1}\dot{\mathrm{U}}_0(\tau) \\ &= w(\tau)\begin{bmatrix} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{bmatrix} \end{aligned}$$

- $\mathscr{H}(\tau)$ is non-diagonalizable matrix.
- Spectrum of $\mathscr{H}(\tau)$ is $\{0\}$.
- $\mathscr{H}(\tau)$ is σ_3-pseudo-normal, i.e., $[\mathscr{H}(\tau), \mathscr{H}(\tau)^\sharp] = 0$, where $\mathscr{H}^\sharp := \sigma_3^{-1}\mathscr{H}^\dagger\sigma_3$.
- $\mathscr{H}(\tau)$ is σ_3-pseudo-Hermitian, if v is real.

- Free particle: $\mathbf{H}(\tau) = -\sigma_3$, $\mathbf{U}(\tau, \tau_0) = \mathbf{U}_0(\tau - \tau_0)$.
- Interaction-picture Hamiltonian: $\Psi(\tau) \to U_0(\tau)^{-1}\Psi(\tau)$

$$\begin{aligned} \mathscr{H}(\tau) &:= \mathbf{U}_0(\tau)^{-1}\mathbf{H}(\tau)\mathbf{U}_0(\tau) - i\mathbf{U}_0(\tau)^{-1}\dot{\mathbf{U}}_0(\tau) \\ &= w(\tau)\begin{bmatrix} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{bmatrix} \end{aligned}$$

Theorem: Let $\mathscr{U}(\tau, \tau_0)$ be the Interaction-picture evolution operator. Then $\mathbf{M} = \mathscr{U}(+\infty, -\infty)$.

Motivation: $\exists$ a dynamical eq. for $\mathscr{U}(\tau, \tau_0)$.

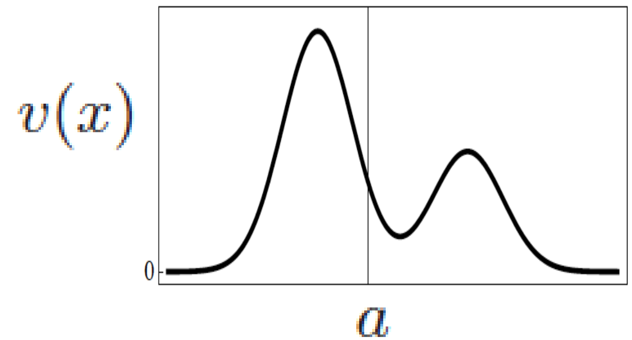
$i\dot{\mathscr{U}}(\tau, \tau_0) = \mathscr{H}(\tau)\mathscr{U}(\tau, \tau_0)$ & $\mathscr{U}(\tau_0, \tau_0) = 1$

Can we find a dynamical eq. for $\mathbf{M}$?

Motivation: $\exists$ a dynamical eq. for $\mathscr{U}(\tau, \tau_0)$.

$i\dot{\mathscr{U}}(\tau, \tau_0) = \mathscr{H}(\tau)\mathscr{U}(\tau, \tau_0)$ & $\mathscr{U}(\tau_0, \tau_0) = 1$

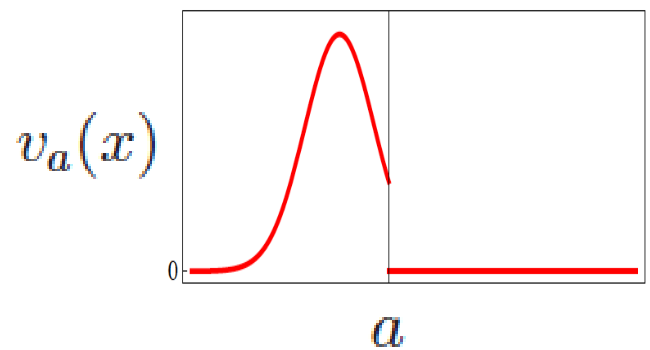
Can we find a dynamical eq. for $\mathbf{M}$?



For each $a \in \mathbb{R}$, let $\alpha := ak$,

$$v_a(x) := \begin{cases} v(x) & \text{for} \quad x \leq a \\ 0 & \text{for} \quad x > a \end{cases},$$

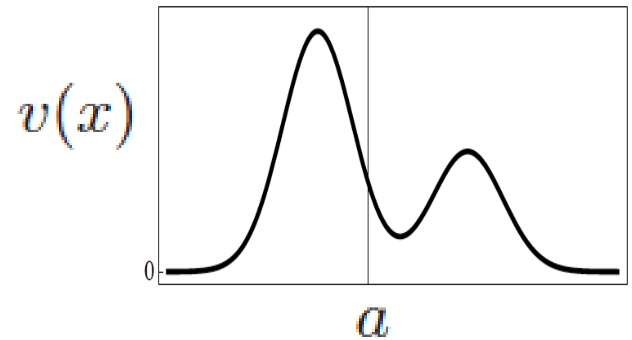
and $\mathbf{M}(\alpha) :=$ transfer matrix of v_a.



Motivation: $\exists$ a dynamical eq. for $\mathscr{U}(\tau, \tau_0)$.

$$i\dot{\mathscr{U}}(\tau,\tau_0) = \mathscr{H}(\tau)\mathscr{U}(\tau,\tau_0) \;\&\; \mathscr{U}(\tau_0,\tau_0) = 1$$

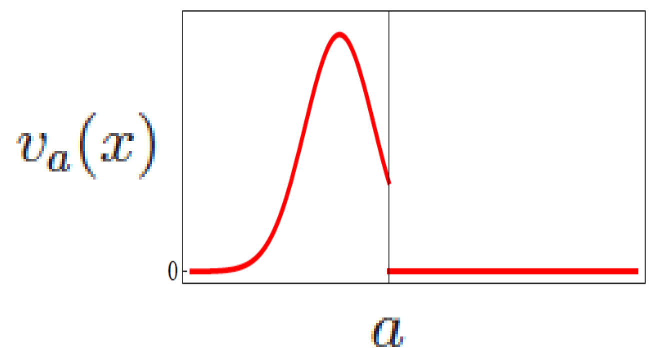
Can we find a dynamical eq. for $\mathbf{M}$?



For each $a \in \mathbb{R}$, let $\alpha := ak$,

$$v_a(x) := \begin{cases} v(x) & \text{for} \quad x \le a \\ 0 & \text{for} \quad x > a \end{cases},$$

and $\mathbf{M}(\alpha) :=$ transfer matrix of v_a.



Then $\mathbf{M}(\alpha) = \mathscr{U}(\alpha, -\infty)$. Therefore,

$$i\partial_\alpha \mathbf{M}(\alpha) = \mathscr{H}(\alpha)\mathbf{M}(\alpha), \qquad \mathbf{M}(-\infty) = 1.$$

We also have $\mathbf{M} = \mathbf{M}(\infty)$.

$$i\partial_\alpha \mathrm{M}(\alpha) = \mathscr{H}(\alpha)\mathrm{M}(\alpha) \text{ \& } \det \mathrm{M} \neq 0 \Rightarrow$$

$$i[\partial_\alpha \mathrm{M}(\alpha)]\mathrm{M}(\alpha)^{-1} = \mathscr{H}(\alpha) = w(\alpha) \begin{bmatrix} 1 & e^{-2i\alpha} \\ -e^{2i\alpha} & -1 \end{bmatrix}$$

Recall $w(\alpha) = v(a)/2k^2$.

$$i\partial_\alpha \mathrm{M}(\alpha) = \mathscr{H}(\alpha)\mathrm{M}(\alpha) \text{ \& } \det \mathrm{M} \neq 0 \Rightarrow$$

$$i[\partial_\alpha \mathrm{M}(\alpha)]\mathrm{M}(\alpha)^{-1} = \mathscr{H}(\alpha) = w(\alpha)\begin{bmatrix} 1 & e^{-2i\alpha} \\ -e^{2i\alpha} & -1 \end{bmatrix}$$

Recall $w(\alpha) = v(a)/2k^2$.

Eg: Square Barrier Potential of height $\mathfrak{z} \in \mathbb{C}$,

$$v(x) = v_L(x) := \begin{cases} \mathfrak{z} & \text{for } x \in [0, L] \\ 0 & \text{for } x \notin [0, L] \end{cases}$$

$$i\partial_\alpha \mathrm{M}(\alpha) = \mathscr{H}(\alpha)\mathrm{M}(\alpha) \ \& \ \det \mathrm{M} \neq 0 \Rightarrow$$

$$i[\partial_\alpha \mathrm{M}(\alpha)]\mathrm{M}(\alpha)^{-1} = \mathscr{H}(\alpha) = w(\alpha) \begin{bmatrix} 1 & e^{-2i\alpha} \\ -e^{2i\alpha} & -1 \end{bmatrix}$$

Recall $w(\alpha) = v(a)/2k^2$.

Eg: Square Barrier Potential of height $\mathfrak{z} \in \mathbb{C}$,

$$v(x) = v_L(x) := \begin{cases} \mathfrak{z} & \text{for } x \in [0, L] \\ 0 & \text{for } x \notin [0, L] \end{cases}$$

$$M_{11}(\alpha) = \left[\cos(\mathfrak{n}\,\alpha) + i(\mathfrak{n}^2 + 1)\sin(\mathfrak{n}\,\alpha)/2\,\mathfrak{n}\right] e^{-i\alpha},$$

$$M_{12}(\alpha) = i(\mathfrak{n}^2 - 1)\sin(\mathfrak{n}\,\alpha)e^{-i\alpha}/2\,\mathfrak{n},$$

$$M_{21}(\alpha) = M_{12}(-\alpha), \qquad M_{22}(\alpha) = M_{11}(-\alpha),$$

$$\mathfrak{n} := \sqrt{1 - \mathfrak{z}/k^2}$$

$$i\partial_\alpha \mathrm{M}(\alpha) = \mathscr{H}(\alpha)\mathrm{M}(\alpha), \qquad \mathrm{M}(-\infty) = 1$$

$$i\partial_\alpha \mathbf{M}(\alpha) = \mathscr{H}(\alpha)\mathbf{M}(\alpha), \qquad \mathbf{M}(-\infty) = 1$$

$$\mathbf{M} = \begin{bmatrix} T - \frac{R^l R^r}{T} & \frac{R^r}{T} \\ -\frac{R^l}{T} & \frac{1}{T} \end{bmatrix}$$

$R^{l/r}(\alpha) := R^{l/r}$ for v_a & $T(\alpha) :=: T$ for $v_a \Rightarrow$

$R^{l/r}(\alpha)$ and $T(\alpha)$ satisfy dynamical eqs.

Finite-range potentials: $w(\tau) = 0$ for $\tau \notin [\tau_-, \tau_+]$.

Finite-range potentials: $w(\tau) = 0$ for $\tau \notin [\tau_-, \tau_+]$.

$$z := e^{-2i\alpha} \in S^1 \subsetneq \mathbb{C} \ \& \ z_\pm := e^{-2i\tau_\pm} \in S^1 \subsetneq \mathbb{C}$$

Finite-range potentials: $w(\tau)=0$ for $\tau\notin[\tau_-,\tau_+]$.

$z := e^{-2i\alpha}\in S^1\subsetneq\mathbb{C}$ & $z_\pm := e^{-2i\tau_\pm}\in S^1\subsetneq\mathbb{C}$

$$\begin{aligned} R^r(z) &= \frac{S(z)}{S'(z)}-z,\\ R^l(z) &= -\int_{z_-}^{z}d\zeta\,\frac{S''(\zeta)}{S(\zeta)S'(\zeta)^2},\\ T(z) &= \frac{1}{S'(z)}. \end{aligned}$$

Finite-range potentials: $w(\tau) = 0$ for $\tau \notin [\tau_-, \tau_+]$.

$z := e^{-2i\alpha} \in S^1 \subsetneq \mathbb{C}$ & $z_\pm := e^{-2i\tau_\pm} \in S^1 \subsetneq \mathbb{C}$

$$\begin{aligned} R^r(z) &= \frac{S(z)}{S'(z)} - z, \\ R^l(z) &= -\int_{z_-}^{z} d\zeta \, \frac{S''(\zeta)}{S(\zeta)S'(\zeta)^2}, \\ T(z) &= \frac{1}{S'(z)}. \end{aligned}$$

$$z^2 S''(z) + \left[\frac{\check{v}(z)}{4k^2}\right] S(z) = 0$$

$$S(z_-) = z_-, \qquad S'(z_-) = 1$$

$$\check{v}(z) := v(i \ln z/2k) = v(a)$$

Finite-range potentials: $w(\tau)=0$ for $x\notin[\tau_-,\tau_+]$.

$z:=e^{-2i\alpha}\in S^1\subsetneq\mathbb{C}$ & $z_\pm:=e^{-2i\tau_\pm}\in S^1\subsetneq\mathbb{C}$

$$\begin{aligned} R^r(z) &= \frac{S(z)}{S'(z)}-z, \\ R^l(z) &= -\int_{z_-}^{z} d\zeta\, \frac{S''(\zeta)}{S(\zeta)S'(\zeta)^2}, \\ T(z) &= \frac{1}{S'(z)}. \end{aligned}$$

$$z^2S''(z)+\left[\frac{\breve{v}(z)}{4k^2}\right]S(z)=0$$

$$S(z_-)=z_-,\qquad S'(z_-)=1$$

$$\breve{v}(z):=v(i\ln z/2k)=v(a)$$

Can use these for inverse scattering.

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

A finite-range potential with a SS at $k = k_0$:

$T = 1/S'$ should have a pole at $k = k_0$.

A. M. Ann. Phys. **341**, 77 (2014)

$$\mathbf{M} = \begin{bmatrix} T - \dfrac{R^l R^r}{T} & \dfrac{R^r}{T} \\ -\dfrac{R^l}{T} & \dfrac{1}{T} \end{bmatrix}$$

A finite-range potential with a SS at $k = k_0$:

$T = 1/S'$ should have a pole at $k = k_0$.

Choose $S(z) = \dfrac{z^2 - 2z_+ z + 1}{2(1 - z_+)}$, $\tau_- = 0$, $\tau_+ = kL$, $k_0 L \notin 2\pi\mathbb{Z}$.

$$\mathfrak{n}^2(x) = 1 - \frac{v(x)}{k^2} = \begin{cases} 1 + 8\left[e^{4ik_0 x} - 2e^{-2ik_0(L-x)} + 1\right]^{-1} & x \in [0, L] \\ 1 & x \notin [0, L] \end{cases}$$

A. M. Ann. Phys. **341**, 77 (2014)

Finite-range right-invisible potential at $k = k_0$:

$R^r = \frac{S}{S'} - z = 0$ & $T = \frac{1}{S'} = 1$ for $z = z_+ = e^{-2i\tau_+}$.

Finite-range right-invisible potential at $k = k_0$:

$R^r = \frac{S}{S'} - z = 0$ & $T = \frac{1}{S'} = 1$ for $z = z_+ = e^{-2i\tau_+}$.

Choose $S(z) = z[\alpha(z-1)^2 + 1]$,

$\tau_- = 0$, $\tau_+ = k_0 L_n := \pi n$, & $n \in \mathbb{Z}^+$:

Finite-range right-invisible potential at $k = k_0$:

$R^r = \frac{S}{S'} - z = 0$ & $T = \frac{1}{S'} = 1$ for $z = z_+ = e^{-2i\tau_+}$.

Choose $S(z) = z[\alpha(z-1)^2 + 1]$,

$\tau_- = 0$, $\tau_+ = k_0 L_n := \pi n$, & $n \in \mathbb{Z}^+$:

$$\mathfrak{n}^2(x) = \begin{cases} 1 + n f_\alpha(x) & x \in [0, L_n] \\ 1 & x \notin [0, L_n] \end{cases}$$

$$v(x) = v_{\alpha,n}(x) := \begin{cases} -k^2 n f_\alpha(x) & x \in [0, L_n] \\ 0 & x \notin [0, L_n] \end{cases}$$

$$f_\alpha(x) := \frac{8\pi i\alpha(3 - 2e^{2ik_0x})}{e^{4ik_0x} + \alpha(1 - e^{2ik_0x})^2}$$

Finite-range right-invisible potential at $k = k_0$:

$R^r = \frac{S}{S'} - z = 0$ & $T = \frac{1}{S'} = 1$ for $z = z_+ = e^{-2i\tau_+}$.

Choose $S(z) = z[\alpha(z-1)^2 + 1]$,

$\tau_- = 0$, $\tau_+ = k_0 L_n := \pi n$, & $n \in \mathbb{Z}^+$:

$$\mathfrak{n}^2(x) = \begin{cases} 1 + n f_\alpha(x) & x \in [0, L_n] \\ 1 & x \notin [0, L_n] \end{cases}$$

$$v(x) = v_{\alpha,n}(x) := \begin{cases} -k^2 n f_\alpha(x) & x \in [0, L_n] \\ 0 & x \notin [0, L_n] \end{cases}$$

$$f_\alpha(x) := \frac{8\pi i\alpha(3 - 2e^{2ik_0x})}{e^{4ik_0x} + \alpha(1 - e^{2ik_0x})^2}$$

At $k = k_0$: $\quad R^r = 0, \quad T = 1$

For $\alpha > -\frac{1}{4}$: $\quad R^l = \frac{-8\pi i n\alpha}{(\alpha + 1)^2}$

$$v_{\alpha,n}(x) := \begin{cases} -k^2 n f_\alpha(x) & x \in [0, L_n] \\ 0 & x \notin [0, L_n] \end{cases}$$

$$f_\alpha(x) := \frac{8\pi i\alpha(3 - 2e^{2ik_0x})}{e^{4ik_0x} + \alpha(1 - e^{2ik_0x})^2} \qquad L_n := \pi n/k_0$$

For given $R = \rho\, e^{i\varphi} \in \mathbb{C}$, choose $\alpha \in [0, 1)$, $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ & $d_m \in \mathbb{R}$, such that

$$\frac{8\pi n\alpha}{(\alpha + 1)^2} = \rho, \qquad d_m = \frac{(4m - 1)\pi - 2\varphi}{4k_0}.$$

Let $v^r_R(x) := v_{\alpha,n}(x + d_m)$ & $v^l_R(x) := v^r_{-R^*}(x)^*$.

$$v_{\alpha,n}(x) := \begin{cases} -k^2 n f_\alpha(x) & x \in [0, L_n] \\ 0 & x \notin [0, L_n] \end{cases}$$

$$f_\alpha(x) := \frac{8\pi i\alpha(3 - 2e^{2ik_0x})}{e^{4ik_0x} + \alpha(1 - e^{2ik_0x})^2} \qquad L_n := \pi n/k_0$$

For given $R = \rho\, e^{i\varphi} \in \mathbb{C}$, choose $\alpha \in [0, 1)$, $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ & $d_m \in \mathbb{R}$, such that

$$\frac{8\pi n\alpha}{(\alpha + 1)^2} = \rho, \qquad d_m = \frac{(4m - 1)\pi - 2\varphi}{4k_0}.$$

Let $v^r_R(x) := v_{\alpha,n}(x + d_m)$ & $v^l_R(x) := v^r_{-R^*}(x)^*$. Then,

- $v^r_R(x)$ is right-invisible with $R^l = R$.
- $v^l_R(x)$ is left-invisible with $R^r = R$.
- Both vanish outside $[-d_m, L_n - d_m]$.

$\Rightarrow$ a **model** for **general unidirectional invisibility**.

Perturbative Expansion for $\mathbf{M}$

$$\mathscr{H}(\tau) = w(\tau) \begin{bmatrix} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{bmatrix} \qquad \tau = kx, \qquad w(\tau) = \frac{v(x)}{2k^2}$$

$$\begin{aligned}
\mathbf{M} &= \mathscr{U}(+\infty, -\infty) = \mathscr{T}\, e^{-i\int_{-\infty}^{\infty} d\tau\, \mathscr{H}(\tau)} \\
&= 1 - i\int_{-\infty}^{\infty} d\tau_1 \mathscr{H}(\tau_1) - \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1 \mathscr{H}(\tau_2)\mathscr{H}(\tau_1) + \cdots \\
&=: 1 + \sum_{\ell=1}^{\infty} \mathbf{M}^{(\ell)}
\end{aligned}$$

Perturbative Expansion for M

$$\mathscr{H}(\tau) = w(\tau)\begin{bmatrix} 1 & e^{-2i\tau} \\ -e^{2i\tau} & -1 \end{bmatrix} \qquad \tau = kx, \qquad w(\tau) = \frac{v(x)}{2k^2}$$

$$\begin{aligned} \mathbf{M} &= \mathscr{U}(+\infty,-\infty) = \mathscr{T}\, e^{-i\int_{-\infty}^{\infty} d\tau\, \mathscr{H}(\tau)} \\ &= 1 - i\int_{-\infty}^{\infty} d\tau_1 \mathscr{H}(\tau_1) - \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1 \mathscr{H}(\tau_2)\mathscr{H}(\tau_1) + \cdots \\ &=: 1 + \sum_{\ell=1}^{\infty} \mathbf{M}^{(\ell)} \end{aligned}$$

$$\mathbf{M}^{(1)} = \frac{-i}{2k}\begin{bmatrix} \tilde{v}(0) & \tilde{v}(2k) \\ -\tilde{v}(-2k) & -\tilde{v}(0) \end{bmatrix},$$

$$\mathbf{M}^{(2)} = \frac{-1}{4k^2}\begin{bmatrix} \tilde{v}(0,0) - \tilde{v}(-2k,2k) & \tilde{v}(2k,0) - \tilde{v}(0,2k) \\ \tilde{v}(-2k,0) - \tilde{v}(0,-2k) & \tilde{v}(0,0) - \tilde{v}(2k,-2k) \end{bmatrix}$$

$$\tilde{f}(k_1,\cdots,k_\ell) := \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_\ell\, e^{-i(k_1x_1+\cdots+k_\ell x_\ell)} f(x_1,\cdots,x_\ell)$$

$$v(x_1,x_2) := v(x_2)\theta(x_2-x_1)v(x_1)$$

Perturbative Unidirectional Invisibility

$$v(x) = \begin{cases} \mathfrak{z}\, f(x) & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L]. \end{cases}$$

$$R^l = \mathcal{O}(\mathfrak{z}^2), \qquad R^r = \mathcal{O}(\mathfrak{z}), \qquad T = 1 + \mathcal{O}(\mathfrak{z}^2).$$

Perturbative Unidirectional Invisibility

$$v(x) = \begin{cases} \mathfrak{z}\, f(x) & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L]. \end{cases}$$

$$R^l = \mathcal{O}(\mathfrak{z}^2), \qquad R^r = \mathcal{O}(\mathfrak{z}), \qquad T = 1 + \mathcal{O}(\mathfrak{z}^2).$$

Example: $f(x) = e^{iKx}$, $k = \dfrac{K}{2} = \dfrac{2\pi m}{L}$, & $m \in \mathbb{Z}^+$.

Perturbative Unidirectional Invisibility

$$v(x) = \begin{cases} \mathfrak{z}\, f(x) & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L]. \end{cases}$$

$R^l = \mathcal{O}(\mathfrak{z}^2)$, $\qquad R^r = \mathcal{O}(\mathfrak{z})$, $\qquad T = 1 + \mathcal{O}(\mathfrak{z}^2)$.

Example: $f(x) = e^{iKx}$, $k = \dfrac{K}{2} = \dfrac{2\pi m}{L}$, & $m \in \mathbb{Z}^+$.

$$R^l = \frac{\tilde{v}(-2k)}{2ik - \tilde{v}(0)} + \mathcal{O}(\mathfrak{z}^2),$$

$$R^r = \frac{\tilde{v}(2k)}{2ik - \tilde{v}(0)} + \mathcal{O}(\mathfrak{z}^2),$$

$$T = \frac{2ik}{2ik - \tilde{v}(0)} + \mathcal{O}(\mathfrak{z}^2).$$

$\Rightarrow$ Complete characterization of pert. unidir. invisibility

⇒ Multimode Perturbative Unidirectional Invisibility

$$v(x) = \begin{cases} \mathfrak{z}\, f(x) & \text{for} \quad x \in [0, L], \\ 0 & \text{for} \quad x \notin [0, L]. \end{cases}$$

Example: $$f(x) = \frac{\mathfrak{a}\, e^{2iKx}}{1 - \mathfrak{a}\, e^{2iKx}} + \frac{\mathfrak{b}\, e^{-iKx}}{1 - \mathfrak{b}\, e^{-2iKx}}$$

$$|\mathfrak{a}| < 1, \qquad |\mathfrak{b}| < 1, \qquad K = \frac{2\pi}{L}$$

Perturbatively invisible from **left**: $k = nK$, $n = 1, 2, 3, \cdots$

Perturbatively invisible from **right**: $k = \left(n + \frac{1}{2}\right) K$.

Perturbative Inverse Scattering:

$$\mathbf{M}^{(1)} = \frac{-i}{2k}\begin{bmatrix} \tilde{v}(0) & \tilde{v}(2k) \\ -\tilde{v}(-2k) & -\tilde{v}(0) \end{bmatrix}$$

Perturbative Inverse Scattering:

$$\mathbf{M}^{(1)} = \frac{-i}{2k}\begin{bmatrix} \tilde{v}(0) & \tilde{v}(2k) \\ -\tilde{v}(-2k) & -\tilde{v}(0) \end{bmatrix}$$

$$v(x) = 2\frac{d}{dx}\check{M}_{12}^{(1)}(2x) = 2\frac{d}{dx}\check{M}_{21}^{(1)}(-2x)$$

$$\check{f}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} f(k)$$

Perturbative Inverse Scattering:

$$\mathbf{M}^{(1)} = \frac{-i}{2k}\begin{bmatrix} \tilde{v}(0) & \tilde{v}(2k) \\ -\tilde{v}(-2k) & -\tilde{v}(0) \end{bmatrix}$$

$$v(x) = 2\frac{d}{dx}\check{M}^{(1)}_{12}(2x) = 2\frac{d}{dx}\check{M}^{(1)}_{21}(-2x)$$

$$\check{f}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} f(k)$$

Theorem: It is the first Born Approximation of the scattering data that determines the form of the potential.

A. M. PRA. **89**, 012709 (2014)

Adiabatic & WKB Approximations

Use adiabatic approx. to solve $i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(t)$:

$$\tau := kx$$

$$w(\tau) := \frac{v(\tau/k)}{2k^2}$$

$$\mathbf{H}(\tau) := \begin{bmatrix} w(\tau) - 1 & w(\tau) \\ -w(\tau) & -w(\tau) + 1 \end{bmatrix}$$

Adiabatic & WKB Approximations

Use adiabatic approx. to solve $i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(t)$:

$$\tau := kx$$

$$w(\tau) := \frac{v(\tau/k)}{2k^2}$$

$$\mathbf{H}(\tau) := \begin{bmatrix} w(\tau) - 1 & w(\tau) \\ -w(\tau) & -w(\tau) + 1 \end{bmatrix}$$

$$\mathbf{H}(\tau)\Psi_{\pm}(\tau) = E_{\pm}(\tau)\Psi_{\pm}(\tau)$$

$$E_{\pm}(\tau) := \pm\mathfrak{n}(\tau)$$

$$\mathfrak{n}(\tau) := \sqrt{1 - \frac{v(\tau/k)}{k^2}}$$

$$\Psi_{\pm}(\tau) := \frac{1}{2}\begin{bmatrix} 1 \mp \mathfrak{n}(\tau) \\ 1 \pm \mathfrak{n}(\tau) \end{bmatrix}$$

Adiabatic & WKB Approximations

Use adiabatic approx. to solve $i\dot{\Psi}(\tau) = \mathbf{H}(\tau)\Psi(t)$:

$$\tau := kx \qquad w(\tau) := \frac{v(\tau/k)}{2k^2} \qquad \mathbf{H}(\tau) := \begin{bmatrix} w(\tau)-1 & w(\tau) \\ -w(\tau) & -w(\tau)+1 \end{bmatrix}$$

$$\mathbf{H}(\tau)\Psi_\pm(\tau) = E_\pm(\tau)\Psi_\pm(\tau)$$

$$E_\pm(\tau) := \pm\mathfrak{n}(\tau) \qquad \Psi_\pm(\tau) := \frac{1}{2}\begin{bmatrix} 1 \mp \mathfrak{n}(\tau) \\ 1 \pm \mathfrak{n}(\tau) \end{bmatrix}$$

$$\mathfrak{n}(\tau) := \sqrt{1 - \frac{v(\tau/k)}{k^2}}$$

Biorthonormal dual of $\Psi_\pm(\tau)$: $\Phi_\pm(\tau) := \frac{1}{2\mathfrak{n}(\tau)^*}\begin{bmatrix} \mathfrak{n}(\tau)^* \mp 1 \\ \mathfrak{n}(\tau)^* \pm 1 \end{bmatrix}$

$$\langle \Phi_i(\tau)|\Psi_j(\tau)\rangle = \delta_{ij}, \qquad \sum_{j=\pm} |\Psi_j(\tau)\rangle\langle\Phi_j(\tau)| = 1$$

$$\mathbf{H}(\tau)\Psi_\pm(\tau) = E_\pm(\tau)\Psi_\pm(\tau)$$

Adiabatic approximation:

$$\Psi_\pm(\tau_0) \longrightarrow \Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

$$\mathbf{H}(\tau)\Psi_\pm(\tau) = E_\pm(\tau)\Psi_\pm(\tau)$$

Adiabatic approximation:

$$\Psi_\pm(\tau_0) \longrightarrow \Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

$$\delta_\pm(\tau) \;=\; -\int_{\tau_0}^{\tau} E_\pm(\tau')d\tau' = \mp \int_{\tau_0}^{\tau} \mathfrak{n}(\tau')d\tau'$$

$$\gamma_\pm(\tau) \;=\; i\int_{\tau_0}^{\tau} \langle \Phi_\pm(\tau')|\dot{\Psi}_\pm(\tau')\rangle d\tau' = i\int_{\mathfrak{n}(\tau_0)}^{\mathfrak{n}(\tau)} \langle \Phi_\pm | d\Psi_\pm \rangle$$

Adiabaticity Condition:

$$\left| \frac{\langle \Phi_\pm(\tau)|\dot{\Psi}_\mp(\tau)\rangle}{E_+(\tau) - E_-(\tau)} \right| \ll 1$$

Garrison & Wright, PLA **128**, 177 (1988)

Adiabaticity Condition:

$$\left|\frac{\langle \Phi_\pm(\tau)|\dot{\Psi}_\mp(\tau)\rangle}{E_+(\tau)-E_-(\tau)}\right| \ll 1 \quad \Leftrightarrow \quad \left|\frac{\dot{\mathfrak{n}}(\tau)}{4\mathfrak{n}(\tau)^2}\right| \ll 1 \quad \Leftrightarrow \quad \frac{|v'(x)|}{8|k^2-v(x)|^{3/2}} \ll 1$$

$$\Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

Adiabaticity Condition:

$$\left|\frac{\langle\Phi_\pm(\tau)|\dot\Psi_\mp(\tau)\rangle}{E_+(\tau)-E_-(\tau)}\right|\ll 1 \quad\Leftrightarrow\quad \left|\frac{\dot{\mathfrak{n}}(\tau)}{4\mathfrak{n}(\tau)^2}\right|\ll 1 \quad\Leftrightarrow\quad \frac{|v'(x)|}{8|k^2-v(x)|^{3/2}}\ll 1$$

$$\Psi(\tau)\approx e^{i\delta_\pm(\tau)}e^{i\gamma_\pm(\tau)}\Psi_\pm(\tau)$$

$$\delta_\pm(\tau)=\mp\int_{\tau_0}^{\tau}\mathfrak{n}(\tau')d\tau'=\mp\int_{x_0}^{x}\sqrt{k^2-v(x')}dx'$$

$$e^{i\gamma_\pm(\tau)}=\sqrt{\frac{\mathfrak{n}(\tau_0)}{\mathfrak{n}(\tau)}} \qquad \Psi_\pm(\tau):=\frac{1}{2}\left[\begin{array}{c}1\mp\mathfrak{n}(\tau)\\ 1\pm\mathfrak{n}(\tau)\end{array}\right]$$

Adiabaticity Condition:

$$\left|\frac{\langle\Phi_\pm(\tau)|\dot{\Psi}_\mp(\tau)\rangle}{E_+(\tau)-E_-(\tau)}\right| \ll 1 \quad\Leftrightarrow\quad \left|\frac{\dot{\mathfrak{n}}(\tau)}{4\mathfrak{n}(\tau)^2}\right| \ll 1 \quad\Leftrightarrow\quad \frac{|v'(x)|}{8|k^2-v(x)|^{3/2}} \ll 1$$

$$\Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

$$\delta_\pm(\tau) = \mp\int_{\tau_0}^{\tau} \mathfrak{n}(\tau')d\tau' = \mp\int_{x_0}^{x}\sqrt{k^2-v(x')}dx'$$

$$e^{i\gamma_\pm(\tau)} = \sqrt{\frac{\mathfrak{n}(\tau_0)}{\mathfrak{n}(\tau)}} \qquad \Psi_\pm(\tau) := \frac{1}{2}\left[\begin{array}{c} 1\mp\mathfrak{n}(\tau) \\ 1\pm\mathfrak{n}(\tau)\end{array}\right]$$

Recall: $\Psi(\tau) := \frac{1}{2}\left[\begin{array}{c}\phi(\tau)-i\dot{\phi}(\tau) \\ \phi(\tau)+i\dot{\phi}(\tau)\end{array}\right]$ & $\phi(\tau) := \psi(\tau/k)$

Adiabaticity Condition:

$$\left|\frac{\langle\Phi_\pm(\tau)|\dot{\Psi}_\mp(\tau)\rangle}{E_+(\tau)-E_-(\tau)}\right| \ll 1 \quad \Leftrightarrow \quad \left|\frac{\dot{\mathfrak{n}}(\tau)}{4\mathfrak{n}(\tau)^2}\right| \ll 1 \quad \Leftrightarrow \quad \frac{|v'(x)|}{8|k^2-v(x)|^{3/2}} \ll 1$$

$$\Psi(\tau) \approx e^{i\delta_\pm(\tau)} e^{i\gamma_\pm(\tau)} \Psi_\pm(\tau)$$

$$\delta_\pm(\tau) = \mp\int_{\tau_0}^{\tau} \mathfrak{n}(\tau')d\tau' = \mp\int_{x_0}^{x}\sqrt{k^2-v(x')}dx'$$

$$e^{i\gamma_\pm(\tau)} = \sqrt{\frac{\mathfrak{n}(\tau_0)}{\mathfrak{n}(\tau)}} \qquad \Psi_\pm(\tau) := \frac{1}{2}\begin{bmatrix} 1\mp\mathfrak{n}(\tau) \\ 1\pm\mathfrak{n}(\tau)\end{bmatrix}$$

Recall: $\Psi(\tau) := \frac{1}{2}\begin{bmatrix}\phi(\tau)-i\dot{\phi}(\tau) \\ \phi(\tau)+i\dot{\phi}(\tau)\end{bmatrix}$ & $\phi(\tau) := \psi(\tau/k)$

$$\psi(x) \approx \sqrt{\frac{\mathfrak{n}(\tau_0)}{\mathfrak{n}(\tau)}}\, e^{\mp i\int_{\tau_0}^{\tau} E_\pm(\tau')d\tau'} = \frac{N_0}{[k^2-v(x)]^{1/4}}\, e^{\mp i\int_{x_0}^{x}\sqrt{k^2-v(x')}dx'}$$

⇒ **WKB Approximation = Adiabatic Approximation**

- **Semiclassical expression for transfer matrix**
- **Higher-order semiclassical scattering**

A. M. JPA 47, 125301 (2014) & 345302 (2014)

Local Inverse Scattering

Problem: Given a positive real number k_0 and complex numbers $R_0^{l/r}$ and T_0 $(\neq 0)$, find a scattering potential $v(x)$ whose reflection and transmission amplitudes at $k = k_0$ are given by $R^{l/r} = R_0^{l/r}$ and $T = T_0$.

Local Inverse Scattering

Problem: Given a positive real number k_0 and complex numbers $R_0^{l/r}$ and T_0 ($\neq 0$), find a scattering potential $v(x)$ whose reflection and transmission amplitudes at $k = k_0$ are given by $R^{l/r} = R_0^{l/r}$ and $T = T_0$.

Solution/Theorem: $v(x)$ can be written as the **sum of at most four unidirectionally invisible finite-range potentials**,

$$v(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x).$$

- $v_i(x)$ have mutually disjoint supports.
- $v_i(x)$ can be selected from the class $\{v_R^r(x), v_R^l(x)\}$.

[A.M., PRA **90**, 023833 (2014)]

Application: Design of **bidirectionally reflectionless phase-shifting amplifier**

Example: Choose $T_0 = \sqrt{2}i$. Then $v_0(x)$ doubles the intensity ($|T_0|^2 = 2$) and produces a $\pi/2$ phase shift in the transmitted wave.

Application: Design of **bidirectionally reflectionless phase-shifting amplifier**

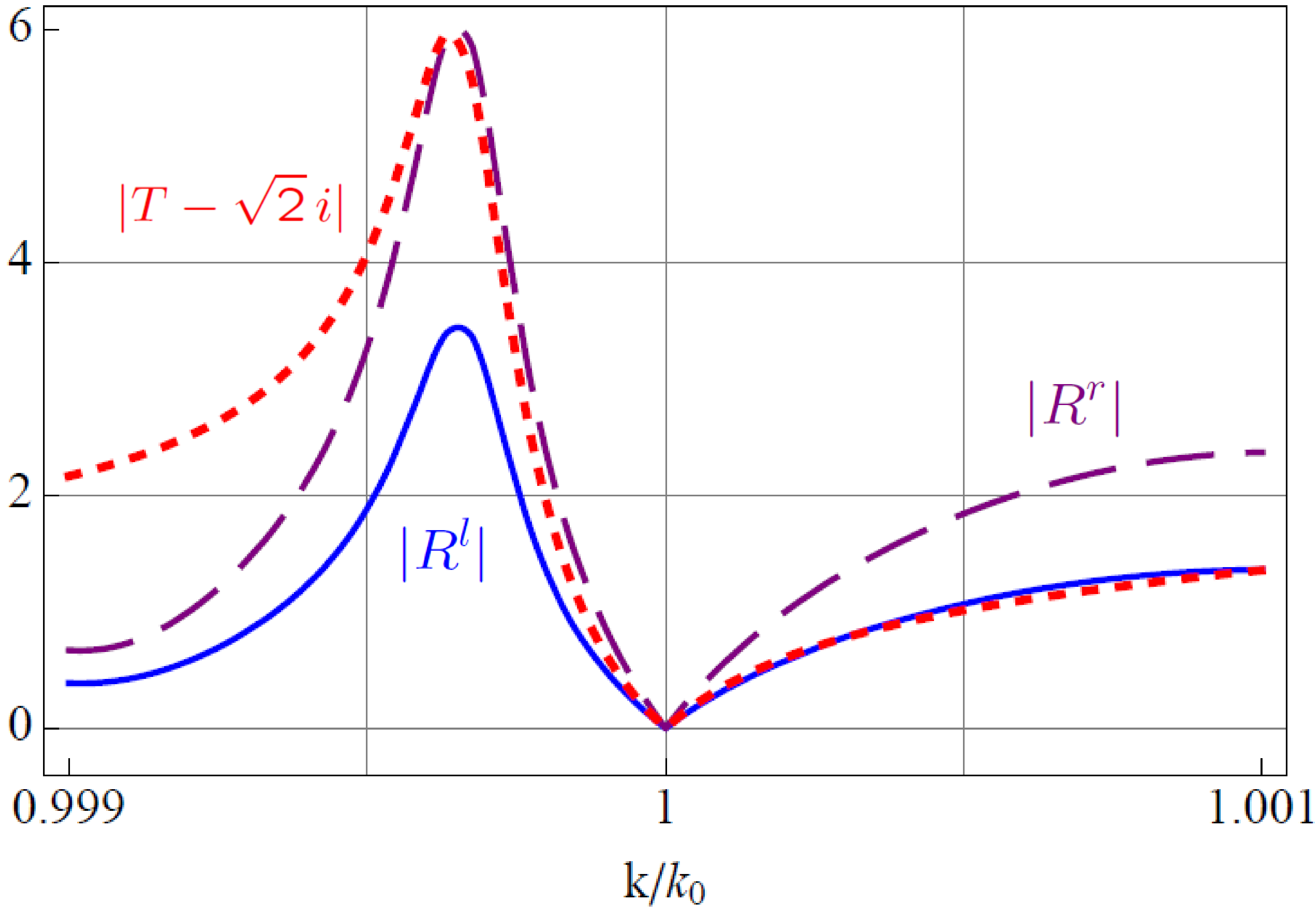
Example: Choose $T_0 = \sqrt{2}i$. Then $v_0(x)$ doubles the intensity ($|T_0|^2 = 2$) and produces a $\pi/2$ phase shift in the transmitted wave.

Explicit model: $v_0(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x)$ with

$$v_j(x) := \begin{cases} v_{\alpha_j,n}(x + d_j) & \text{for} \quad j = 1, 3, \\ v_{\alpha_j,n}(x + d_j)^* & \text{for} \quad j = 2, 4, \end{cases}$$

$k_0 = 2\pi/\mu\text{m}$, $n = 300$, so that $L_n = 150\ \mu\text{m}$, and

$$\begin{aligned} \alpha_1 &= 1.57798 \times 10^{-4}, & d_1 &= 300.625\ \mu\text{m}, \\ \alpha_2 &= 1.93283 \times 10^{-4}, & d_2 &= 150.299\ \mu\text{m}, \\ \alpha_3 &= 1.11565 \times 10^{-4}, & d_3 &= 0.00000\ \mu\text{m}, \\ \alpha_4 &= 2.73409 \times 10^{-4}, & d_4 &= -150.326\ \mu\text{m}. \end{aligned}$$



6
4
2
0
0.999
1
1.001
k/k_0
$|T - \sqrt{2}\, i|$
$|R^l|$
$|R^r|$

Summary:

- Pseudo-Hermitian QM: Spectral singularities appear as singularities of the metric operator for complex scattering potentials.

- Physically spectral singularities correspond to the scattering states with real and positive energy that behave exactly like a zero-width resonance. In optics they appear as lasing at threshold gain. Their time-reversal gives rise to anti-lasing.

- $\mathbf{M}$ = S-matrix for a two-level non-Hermitian Hamiltonian which is pseudo-Hermitian for a real potential.

- $\mathbf{M}$ = Asymptotic value of the evolution operator for a two-level pseudo-normal Hamiltonian.

- Dynamical equations for $\mathbf{M}$ $\Rightarrow$ optical potential design

- Perturbative Unidirectional Invisibility & inverse scattering

- Adiabatic approximation $\Leftrightarrow$ WKB approximation

- Pre-exponential part of the WKB wave functions is actually a complex geometric phase

- **Explicit model** for unidirectional invisibility
- Unidirectionally invisible potentials are **local building blocks** of all scattering potentials
- **Applications:** Design of reflectionless amplifiers, absorbers, phase-shifters, threshold lasers & anti-lasers.

References:

- arXiv:1310.0592 [Ann. Phys. (NY), **341**, 77 (2014)]
- arXiv:1311.1619 [Phys. Rev. A **89**, 012709 (2014)]
- arXiv:1401.4315 [J. Phys. A **47**, 125301 (2014)]
- arXiv:1402.6458 [J. Phys. A, **47**, 345302 (2014)]
- arXiv:1407.1760 [Phys. Rev. A, **90**, 023833 (2014)]

- **Explicit model** for unidirectional invisibility
- Unidirectionally invisible potentials are **local building blocks** of all scattering potentials
- **Applications:** Design of reflectionless amplifiers, absorbers, phase-shifters, threshold lasers & anti-lasers.

References:

- arXiv:1310.0592 [Ann. Phys. (NY), **341**, 77 (2014)]
- arXiv:1311.1619 [Phys. Rev. A **89**, 012709 (2014)]
- arXiv:1401.4315 [J. Phys. A **47**, 125301 (2014)]
- arXiv:1402.6458 [J. Phys. A, **47**, 345302 (2014)]
- arXiv:1407.1760 [Phys. Rev. A, **90**, 023833 (2014)]

Thank you for your attention.