1

Analysis over supernumbers

1.1 Supernumbers and superanalytic functions

Grassmann algebras

Let ζ^a, $a = 1, \ldots N$, be a set of generators for an algebra, which anticommute:

$$\zeta^a \zeta^b = -\zeta^b \zeta^a, \qquad (\zeta^a)^2 = 0, \quad \text{for all } a, b. \tag{1.1.1}$$

The algebra is called a *Grassmann algebra* and will be denoted by Λ_N. We shall usually, though not always, deal with the formal limit $N \to \infty$. The corresponding algebra will be denoted by Λ_∞.

The elements $1, \zeta^a, \zeta^a\zeta^b, \ldots$, where the indices in each product are all different, form an infinite basis for Λ_∞. When N is finite the sequence terminates at $\zeta^1 \ldots \zeta^N$ and there are only 2^N distinct basis elements. Under addition as well as multiplication by a complex number, the elements of Λ_N form a linear vector space of 2^N dimensions; the elements of Λ_∞ form an infinite-dimensional vector space. As algebras over the complex numbers (which is the only field we shall consider) Λ_N and Λ_∞ are associative but not commutative (excluding the trivial cases $N = 0, 1$).

Supernumbers

The elements of Λ_∞ will be called *supernumbers*. Every supernumber can be expressed in the form

$$z = z_B + z_S, \tag{1.1.2}$$

where z_B is an ordinary complex number and

$$z_S = \sum_{n=1}^{\infty} \frac{1}{n!} c_{a_1 \ldots a_n} \zeta^{a_n} \ldots \zeta^{a_1}, \tag{1.1.3}$$

the c's also being complex numbers. The c's are completely antisymmetric in their indices, and summation over repeated indices is to be understood unless otherwise stated. The number z_B will be called the *body* and the remainder z_S will be called the *soul* of the supernumber z. If Λ_∞ is replaced by Λ_N (N finite) then the soul of a supernumber is always nilpotent:

$$z_S^{N+1} = 0. \tag{1.1.4}$$

When N is infinite the soul need not be nilpotent.

When N is finite the condition $\zeta^a z = 0$ for all a implies that z has the form $z = c\zeta^1 \ldots \zeta^N$ for some complex number c. When N is infinite the condition $\zeta^a z = 0$ for all a implies $z = 0$.

A supernumber has an inverse if and only if its body is nonvanishing. The inverse, which is unique, is given by the formula

$$z^{-1} = z_B^{-1} \sum_{n=0}^{\infty} (-z_B^{-1} z_S)^n. \tag{1.1.5}$$

Series of this kind may be introduced to extend any analytic function f on the complex numbers to a supernumber-valued function on Λ_∞:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_B) z_S^n. \tag{1.1.6}$$

Here $f^{(n)}(z_B)$ denotes the nth derivative of f at the point z_B in the complex plane, and the definition is valid for all z_B that are not singular points of f. Because of (1.1.4) the 'Taylor series' (1.1.6) terminates when N is finite. By substituting (1.1.3) into (1.1.6) one can obtain an expansion of $f(z)$ in terms of the basis elements of Λ_∞. When N is infinite expressions (1.1.5) and (1.1.6), as well as their expansions in terms of basis elements, are formal infinite series. The coefficient of each term is unique and finite.

One may consider matrices whose elements are supernumbers. The body of a matrix is then defined as the ordinary matrix obtained by replacing each element with its body. The soul of a matrix is the remainder. A square matrix has an inverse, and is said to be *nonsingular*, if and only if its body is nonsingular. The inverse is unique and is given by a formula analogous to (1.1.5).

c-numbers and a-numbers

Any supernumber may be split into its even and odd parts:

$$z = u + v, \tag{1.1.7}$$

$$u = z_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{a_1 \ldots a_{2n}} \zeta^{a_{2n}} \ldots \zeta^{a_1}, \tag{1.1.8}$$

$$v = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{a_1 \ldots a_{2n+1}} \zeta^{a_{2n+1}} \ldots \zeta^{a_1}. \tag{1.1.9}$$

purely odd or purely even. Odd supernumbers anticommute among themselves and will be called *a-numbers.* Even supernumbers commute with everything and will be called *c-numbers.* A-numbers possess no body and hence are not invertible. The set of all c-numbers is a commutative subalgebra of Λ_∞, which will be denoted by $\mathbf{C}_c$. The set of all a-numbers will be denoted by $\mathbf{C}_a$; it is *not* a subalgebra. The product of two c-numbers, or of two a-numbers, is a c-number. The product of an a-number and a c-number is an a-number. The square of every a-number vanishes.

If Λ_∞ is replaced by Λ_N, $\mathbf{C}_c$ and $\mathbf{C}_a$ become 2^{N-1}-dimensional vector spaces. In the formal limit $N \to \infty$ they may continue to be regarded as vector spaces, but we shall not give them a norm or even a topology.

Superanalytic functions of supernumbers

Just as ordinary analysis can be constructed as the theory of analytic mappings of the complex plane into itself, so can an analytic theory of functions of c-numbers and a-numbers be built up by studying mappings from $\mathbf{C}_c$ or $\mathbf{C}_a$ to Λ_∞.

Consider first $\mathbf{C}_a$. Let Λ_∞ for the moment be replaced by Λ_N and let f be a mapping from $\mathbf{C}_a$ into Λ_N. Since, when N is finite, $\mathbf{C}_a$ and Λ_N are finite-dimensional vector spaces over the complex numbers, one has a body of conventional theory on which to draw in order to define, for example, the condition that f be a differentiable mapping at a point v of $\mathbf{C}_a$. Mere differentiability, however, does not involve the algebraic structure of $\mathbf{C}_a$ and Λ_N. What is more interesting is to pass immediately to the formal limit $N \to \infty$ and to demand that f be *superanalytic* at that point. By this is meant the following: let v be given an arbitrary infinitesimal a-number displacement $\mathrm{d}v$. Then its image $f(v)$ in Λ_∞ must suffer a displacement which, for all $\mathrm{d}v$. takes the form

$$\mathrm{d}f(v) = \mathrm{d}v\left[\frac{\vec{\mathrm{d}}}{\mathrm{d}v}f(v)\right] = \left[f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v}\right]\mathrm{d}v, \tag{1.1.10}$$

where the coefficients $\dfrac{\vec{\mathrm{d}}}{\mathrm{d}v}f(v)$ and $f(v)\dfrac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v}$ are independent of $\mathrm{d}v$ and depend (at most) only on v. These coefficients are called, respectively, the *left* and *right derivatives* of f with respect to v.

It can be shown (see exercise **1.1**) that the general solution of eq. (1.1.10) is

$$f(v) = a + bv, \tag{1.1.11}$$

where a and b are arbitrary constant elements of Λ_∞. That is, a superanalytic function of an a-number variable is simply a linear function! It is therefore superanalytic everywhere in $\mathbf{C}_a$ (no singularities). If the coefficient b is separated into its even and odd parts,

$$b = b_{\mathrm{e}} + b_{\mathrm{o}}, \tag{1.1.12}$$

then one may write

$$f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v} = b_{\mathrm{e}} + b_{\mathrm{o}}, \quad \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v) = b_{\mathrm{e}} - b_{\mathrm{o}}, \tag{1.1.13}$$

and one sees that the left and right derivatives of f are, in fact, constants, independent of v, a fact that may be expressed in the form

$$f(v)\frac{\overleftarrow{\mathrm{d}}\,\overleftarrow{\mathrm{d}}}{\mathrm{d}v\,\mathrm{d}v} = \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v} = \frac{\overrightarrow{\mathrm{d}}\,\overrightarrow{\mathrm{d}}}{\mathrm{d}v\,\mathrm{d}v}f(v) = 0. \tag{1.1.14}$$

If the range of f is contained in $\mathbf{C}_c$ then a is even, b is odd, and $\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v) = -f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v}$. If, on the other hand, the range of f is contained in $\mathbf{C}_a$ then a is odd, b is even and $\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v) = f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v}$.

Superanalytic mappings f from $\mathbf{C}_c$ to Λ_∞ are defined similarly:

$$\mathrm{d}f(u) = \mathrm{d}u\left[\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}u}f(u)\right] = \left[f(u)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}u}\right]\mathrm{d}u. \tag{1.1.15}$$

But here the similarity ends. Because $\mathrm{d}u$ is a c-number it follows that

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}u}f(u) = f(u)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}u}, \tag{1.1.16}$$

so there is no need to distinguish between right and left derivatives. Moreover, the class of superanalytic functions of a c-number variable is infinitely richer than the class represented by eq. (1.1.11). For example, for every ordinary analytic function f on the complex numbers there is a superanalytic function over $\mathbf{C}_c$ analogous to (1.1.6):

$$f(u) \stackrel{\mathrm{def}}{=} \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(u_{\mathrm{B}})u_{\mathrm{S}}^{n}, \tag{1.1.17}$$

where u_{B} and u_{S} are respectively the body and soul of u. The general solution of eq. (1.1.15) has the form

$$f(u) = \sum_{n=0}^{\infty}\frac{1}{n!}f_{a_1\ldots a_n}(u)\zeta^{a_n}\ldots\zeta^{a_1}, \tag{1.1.18}$$

where the $f_{a_1\ldots a_n}(u)$ are functions like (1.1.17). If the range of f is contained in $\mathbf{C}_c$ then the $f_{a_1\ldots a_n}$, with n odd, vanish. If the range is contained in $\mathbf{C}_a$ then the $f_{a_1\ldots a_n}$, with n even, vanish.

Integration of superanalytic functions of supernumbers

The theory of integration too may be generalized from the ordinary complex plane $\mathbf{C}$ to the spaces $\mathbf{C}_c$ and $\mathbf{C}_a$. Consider first $\mathbf{C}_c$. The singularities (poles, branch points, etc.) of the functions (1.1.17) and (1.1.18) are located at specific values of u_B, independent of u_S. Therefore, in speaking of the location of these singularities relative to a given curve in $\mathbf{C}_c$, one may always imagine the curve to be projected onto the u_B plane and use such conventional terms as 'left', 'right', 'above', 'below', 'inside', 'on', etc. Any line integral of the form $\int_{u_1}^{u_2} f(u)\,du$ depends only on the endpoints and on the homotopic relation of the curve to the various singularities of f, and not on the specific curve. If f is superanalytic on and inside a *closed* curve in $\mathbf{C}_c$ then the curve may be continuously deformed to a point without crossing any singularity and

$$\oint f(u)\,du = 0. \tag{1.1.19}$$

More generally, if f is superanalytic on the curve and superanalytic inside except at a finite number of poles, then

$$\oint f(u)\,du = 2\pi\mathrm{i} \times (\text{sum of residues at the poles}). \tag{1.1.20}$$

Note that if f has the general form (1.1.18) the residues may be arbitrary supernumbers.

Line integrals of superanalytic functions over $\mathbf{C}_a$ do not behave analogously. Integrals $\oint f(v)\,dv$ over closed curves in $\mathbf{C}_a$ do *not* vanish unless $f(v)$ is itself the derivative of a superanalytic function, i.e., a constant (see eqs. (1.1.13) and (1.1.14)). In particular the integral $\oint v\,dv$ depends in a continuous fashion on the contour. We shall presently wish to attach an alternative meaning to the symbol $\int v\,dv$, for which there is no such ambiguity. But before doing so we need to take a look at functions of a real variable.

1.2 Real supernumbers. Differentiable functions of real *c*-numbers, and their integrals

Complex conjugation

In order to define *real* supernumbers we have to make some rules about complex conjugation (denoted here by an asterisk*). The laws of complex conjugation of sums and products of supernumbers will be taken in the form

$$(z+z')^* = z^* + z'^*, \qquad (zz')^* = z'^* z^*, \quad \text{for all } z, z' \text{ in } \Lambda_\infty. \quad (1.2.1)$$

The complex conjugate of the body of a supernumber will be taken to be its ordinary complex conjugate, and the generators of Λ_∞ will be assumed to be 'real':

$$\zeta^{a*} = \zeta^a, \quad \text{for all } a. \qquad (1.2.2)$$

Evidently

$$(\zeta^a \zeta^b \ldots \zeta^c)^* = \zeta^c \ldots \zeta^b \zeta^a, \qquad (1.2.3)$$

and from this, together with the anticommutation law (1.1.1), one may infer that the basis element $\zeta^{a_1} \ldots \zeta^{a_n}$ is real when $\frac{1}{2}n(n-1)$ is even and imaginary when $\frac{1}{2}n(n-1)$ is odd. (As for ordinary numbers, a supernumber z is said to be *real* if $z^* = z$ and *imaginary* if $z^* = -z$.) A general element of Λ_∞ is real if and only if both its body and soul are real. The soul will be real if and only if the coefficients $c_{a_1 \ldots a_n}$ in the expansion (1.1.3) are real when $\frac{1}{2}n(n-1)$ is even and imaginary when $\frac{1}{2}n(n-1)$ is odd.

We shall denote by $\mathbf{R}_c$ the subset of all real elements of $\mathbf{C}_c$ and by $\mathbf{R}_a$ the subset of all real elements of $\mathbf{C}_a$. The set $\mathbf{R}_c$ is a subalgebra of $\mathbf{C}_c$. The product of two real c-numbers is a real c-number. The product of a real c-number and a real a-number is a real a-number. The product of two real a-numbers is an *imaginary* c-number.

The symbol 'x' will generally be used to denote a real variable, whether over $\mathbf{R}_c$ or over $\mathbf{R}_a$. When it is necessary to emphasize which of the two domain spaces is relevant we shall sometimes revert to using the symbols 'u' and 'v', with the understanding that their values are restricted to be real.

Functions, distributions and integrals over $\mathbf{R}_c$

A function from $\mathbf{R}_c$ to Λ_∞ need *not* be the restriction to $\mathbf{R}_c$ of a superanalytic function over $\mathbf{C}_c$ in order to be differentiable in the sense of eq. (1.1.15) *with* du *now restricted to* $\mathbf{R}_c$. Let f be any C^∞ function of an ordinary real variable. It may be generalized to a differentiable function over $\mathbf{R}_c$ in complete analogy with eq. (1.1.17):

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_{\mathrm{B}}) x_{\mathrm{S}}^n. \qquad (1.2.4)$$

If Λ_∞ is replaced by Λ_N the series (1.2.4) terminates at $n = [N/2]$, and f need be only $C^{[N/2]+1}$. Note that in the limit $N \to \infty$ there is no convergence problem, for (1.2.4) is a formal series.

Instead of starting with a C^∞ function one may equally well start with an arbitrary *distribution*. Since the notion of the derivative of a distribution is well defined, eq. (1.2.4) may then be regarded as defining a *distribution* over $\mathbf{R}_c$. The presence of a soul in the independent variable evidently has little practical effect on the variety of functions with which one may work in applications of the theory. In this respect $\mathbf{R}_c$ is a harmless generalization of its own subspace $\mathbf{R}$, the real line.

Consider now a contour that is restricted to lie wholly within $\mathbf{R}_c$. The value of the integral, over this contour, of a function f defined over $\mathbf{R}_c$, will depend only on the endpoints of the contour, provided merely that f is differentiable in the sense of eq. (1.1.15) with $\mathrm{d}u$ restricted to $\mathbf{R}_c$. That is, f *does not have to be analytic.* A schematic proof of this fact may be constructed along the lines of the suggested proof for exercise **1.2**, in the case in which f, when restricted to $\mathbf{R}$, is C^∞. More generally, suppose that f is a distribution possessing only a discrete set of singular points (describable, for example, in terms of delta functions and their derivatives). Let $F(x)$ be the corresponding generalization to $\mathbf{R}_c$ of the indefinite integral $\int f(x)\,\mathrm{d}x$. Then taking, for simplicity, the case of a contour between two points a and b in $\mathbf{R}_c$ for which x_S is a smooth single-valued function of x_B, one may write

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{a_\mathrm{B}}^{b_\mathrm{B}} f^{(n)}(x_\mathrm{B})x_\mathrm{S}^n(x_\mathrm{B})[1+x_\mathrm{S}'(x_\mathrm{B})]\,\mathrm{d}x_\mathrm{B}\\
&= \sum_{n=0}^{\infty}\int_{a_\mathrm{B}}^{b_\mathrm{B}} f^{(n)}(x_\mathrm{B})\left[\frac{1}{n!}x_\mathrm{S}^n(x_\mathrm{B}) + \frac{1}{(n+1)!}\frac{\mathrm{d}}{\mathrm{d}x_\mathrm{B}}x_\mathrm{S}^{n+1}(x_\mathrm{B})\right]\mathrm{d}x_\mathrm{B}\\
&= \int_{a_\mathrm{B}}^{b_\mathrm{B}} f[1+x_\mathrm{S}'(x_\mathrm{B})]\,\mathrm{d}x_\mathrm{B}\\
&\quad + \sum_{n=1}^{\infty}\frac{1}{n!}[f^{(n-1)}(b_\mathrm{B})b_\mathrm{S}^n - f^{(n-1)}(a_\mathrm{B})a_\mathrm{S}^n]\\
&\quad + \sum_{n=1}^{\infty}\int_{a_\mathrm{B}}^{b_\mathrm{B}}\left[-\frac{1}{n!}f^{(n-1)}(x_\mathrm{B})\frac{\mathrm{d}}{\mathrm{d}x_\mathrm{B}}x_\mathrm{S}^n(x_\mathrm{B})\right.\\
&\qquad\qquad \left. + \frac{1}{(n+1)!}f^{(n)}(x_\mathrm{B})\frac{\mathrm{d}}{\mathrm{d}x_\mathrm{B}}x_\mathrm{S}^{n+1}(x_\mathrm{B})\right]\mathrm{d}x_\mathrm{B}\\
&= F(b) - F(a), \qquad (1.2.5)\end{aligned}$$

provided neither a_B nor b_B lies on a singular point of any of the $f^{(n)}$. The derivation may easily be generalized to include contours for which the soul is a multi-valued function of the body.

Fourier transforms over $\mathbf{R}_c$

The contour independence of the above result implies that in working with integrals over $\mathbf{R}_c$ one may for many purposes proceed as if one were working over $\mathbf{R}$. A striking illustration of this is provided by the theory of Fourier transforms, which remains totally unchanged in form under the generalization from $\mathbf{R}$ to $\mathbf{R}_c$. The essence of this theory is summed up in the formula

$$\delta(x) = (2\pi)^{-1}\int_{\mathbf{R}_c} e^{ipx}\,dp \overset{\text{def}}{=} \lim_{\varepsilon\to+0}(2\pi)^{-1}\int_{\mathbf{R}_c} e^{ipx-\varepsilon p^2}\,dp, \qquad (1.2.6)$$

where the limit is understood to be taken after all integrations involving $\delta(x)$ have been performed. The symbol '$\int_{\mathbf{R}_c}$' means 'integrate over any contour in $\mathbf{R}_c$ the bodies of whose endpoints tend to $-\infty$ and $+\infty$ respectively'. How the soul behaves along the contour is completely irrelevant. Because the integrand is an entire function that vanishes at the endpoints independently of their souls, the contour may be displaced until it coincides with $\mathbf{R}$, without affecting the value of the integral. All the usual theory thereupon applies. It applies, in fact, even if x itself possesses a soul (!), for (1.2.6) then implies

$$\delta(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\delta^{(n)}(x_{\mathrm{B}})x_{\mathrm{S}}^{n}, \qquad (1.2.7)$$

and if f is any function of the form (1.2.4) where none of the $f^{(n)}$ are singular at $x_{\mathrm{B}} = 0$, then its product with $\delta(x)$ is a function of the form to which eq. (1.2.5) applies, which means that $\int_C f(x)\delta(x)\,dx$ depends only on the endpoints of the contour C. If the bodies of the endpoints are $-\infty$ and $+\infty$ respectively, then the integrand vanishes at these endpoints, and the contour may be displaced until it coincides with $\mathbf{R}$. This implies

$$\int_{\mathbf{R}_c} f(x)\delta(x)\,dx = f(0), \qquad (1.2.8)$$

even if the contour does not pass through the point $x = 0$! From now on we shall often omit the subscript $\mathbf{R}_c$ on the symbol '$\int_{\mathbf{R}_c}$', in analogy with the frequent custom of omitting the $\pm\infty$ on '$\int_{-\infty}^{\infty}$'.

1.3 Functions and integrals over $\mathbf{R}_a$

Basic definitions

To develop an integration theory for functions on $\mathbf{R}_a$, one must proceed rather differently, indeed in a manner that appears, at first sight, bizarre.

By so doing one can obtain a theory that displays remarkable analogies to integration theory over $\mathbf{R}_c$. The first thing that one must do is give up the idea that a definite integral is associated with a family of paths all having the same, or related, endpoints. We have seen, in any case, that such integrals are path independent only in the trivial case in which the integrand is a constant.

We shall confine our attention to functions differentiable in the sense of eq. (1.10) with $\mathrm{d}v$ now restricted to $\mathbf{R}_a$ and v replaced by x. All such functions have the form (cf. eq. (1.1.11))

$$f(x) = a + bx, \tag{1.3.1}$$

where a and b are constant supernumbers.[†] Because (1.3.1) is a linear form in x, in order to give a meaning to the symbol '$\int f(x)\,\mathrm{d}x$', one has only to decide what meaning to give to the symbols '$\int \mathrm{d}x$' and '$\int x\,\mathrm{d}x$'. The integrals of all other functions will then be determined by the rules

$$\int [f(x) + g(x)]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x, \quad \text{for all } f \text{ and } g \text{ on } \mathbf{R}_a{}^{\ddagger} \tag{1.3.2}$$

$$\int af(x)\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x, \quad \text{for all } a \text{ in } \Lambda_\infty \text{ and all } f \text{ on } \mathbf{R}_a. \tag{1.3.3}$$

In choosing the basic integrals we shall be guided by analogy with the equation

$$\int \left[\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right]\mathrm{d}x = 0, \tag{1.3.4}$$

which holds for differentiable functions (or distributions) $f(x)$ on $\mathbf{R}_c$ satisfying $f(x) \xrightarrow[x_B \to \pm\infty]{} 0$. If we require eq. (1.3.4) to hold *also* on $\mathbf{R}_a$ then we must necessarily have

$$\int \mathrm{d}x \stackrel{\text{def}}{=} 0, \tag{1.3.5}$$

$$\int x\,\mathrm{d}x \stackrel{\text{def}}{=} Z, \tag{1.3.6}$$

where Z is some constant supernumber.

In order to accept the definitions (1.3.5) and (1.3.6) the reader must give

[†] If f takes its values in $\mathbf{R}_a$ then a is a real a-number and b a real c-number. If f takes its values in $\mathbf{R}_c$ then a is a real c-number and b is an imaginary a-number.

[‡] Here, and in what follows, by the phrase 'all f on $\mathbf{R}_a$' is meant 'all f of the form (1.3.1) with x in $\mathbf{R}_a$'.

up conventional prejudices. Measure-theoretical notions play no role here. Integration over $\mathbf{R}_a$ becomes a purely formal procedure, the utility of which rests ultimately on the naturalness with which it can be used to encode certain algebraic information. One may note already that eqs. (1.3.2), (1.3.3), (1.3.5) and (1.3.6) together imply the law of shifting the integration variable and the law of integration by parts:

$$\int f(x+a)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x, \qquad (1.3.7)$$

$$\int f(x)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}x}g(x)\,\mathrm{d}x = \int f(x)\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}g(x)\,\mathrm{d}x, \qquad (1.3.8)$$

for all f and g on $\mathbf{R}_a$ and all a in $\mathbf{R}_a$. The proofs of eqs. (1.3.7) and (1.3.8) are easy exercises. If f takes its values in $\mathbf{R}_c$ then (1.3.8) may be rewritten in the more familiar form

$$\int f(x)\left[\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}g(x)\right]\mathrm{d}x = -\int\left[\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}f(x)\right]g(x)\,\mathrm{d}x.$$

Equation (1.3.6) will be supplemented by the convention

$$\int x\,\mathrm{d}x = -\int \mathrm{d}x\,x. \qquad (1.3.9)$$

That is, the symbol '$\mathrm{d}x$' will be treated formally as if $\mathrm{d}x$ were an a-number. It is not, however, to be imagined as being, like x, a real a-number; nor is the formal bodilessness of $\mathrm{d}x$ to be regarded as implying that the constant Z in eq. (1.3.6) has vanishing body. The only condition that the anticommutativity of $\mathrm{d}x$ imposes is that Z be a c-number.

Fourier transforms over $\mathbf{R}_a$

We shall fix Z by drawing on another analogy (in addition to eq. (1.3.4)) with integration theory over $\mathbf{R}_c$. First note that if Z has nonvanishing body then an analog for the delta function exists for integrals over $\mathbf{R}_a$, namely

$$\delta(x) = Z^{-1}x. \qquad (1.3.10)$$

As may be readily verified this function satisfies

$$\int f(x)\delta(x)\,\mathrm{d}x = f(0), \quad \text{for all } f \text{ on } \mathbf{R}_a, \qquad (1.3.11)$$

the order of factors in the integrand being important. It is a remarkable fact that if Z is chosen appropriately this delta function may be expressed as a 'Fourier integral', in complete analogy with (1.2.6):

$$\delta(x) = (2\pi)^{-1} \int e^{ipx} \, dp. \tag{1.3.12}$$

Here p is an a-number and the integrand is to be regarded as a function on $\mathbf{R}_a \times \mathbf{R}_a$.[†] Since the series for the exponential terminates, the integral is readily evaluated:

$$(2\pi)^{-1} \int e^{ipx} \, dp = (2\pi)^{-1} \int (1 + ipx) \, dp = -(2\pi)^{-1} ix \int p \, dp$$
$$= (2\pi i)^{-1} Zx. \tag{1.3.13}$$

Equating expressions (1.3.10) and (1.3.13) one infers

$$Z^2 = 2\pi i. \tag{1.3.14}$$

The phase of Z will be fixed by the convention

$$Z = (2\pi i)^{\frac{1}{2}} = (2\pi)^{\frac{1}{2}} e^{\pi i/4}. \tag{1.3.15}$$

It will be noted that, contrary to the situation on $\mathbf{R}_c$, the delta function on $\mathbf{R}_a$ is an odd function of its argument: $\delta(-x) = -\delta(x)$. A related fact is that the order of p and x in the exponent of (1.3.12) is important, and a choice of order has to be made. It is perfectly possible to develop the theory with the opposite ordering, but then the sign in front of i in many of the previous equations must be changed.

With the delta function expressible in the form (1.3.12) a theory of Fourier transforms may be developed. The Fourier transform of a function f on $\mathbf{R}_a$ will be defined by

$$\tilde{f}(p) \stackrel{\text{def}}{=} (2\pi)^{-\frac{1}{2}} \int f(x) e^{ixp} \, dx, \tag{1.3.16}$$

the order of factors being, as usual, important. If f has the form (1.3.1) then its Fourier transform is readily computed to be

$$\tilde{f}(p) = i^{\frac{1}{2}} b + i^{-\frac{1}{2}} ap, \tag{1.3.17}$$

from which one immediately sees that the original function is regained by taking the Fourier transform twice.[‡] Another proof of this is as follows:

$$\tilde{\tilde{f}}(x) = (2\pi)^{-\frac{1}{2}} \int \tilde{f}(p) e^{ipx} \, dp$$
$$= (2\pi)^{-1} \int\int f(x') e^{ix'p} \, dx' e^{ipx} \, dp$$

[†] Since px is imaginary the integrand takes its values in $\mathbf{R}_c$.

[‡] Note that no intervening operation of reflection in the origin ($x \to -x$) is required as in the case of ordinary Fourier transforms on $\mathbf{R}_c$.

$$= -(2\pi)^{-1}\iint f(x')\mathrm{e}^{-\mathrm{i}p(x'-x)}\,\mathrm{d}p\,\mathrm{d}x'$$

$$= \int f(x')\delta(x'-x)\,\mathrm{d}x' = f(x). \tag{1.3.18}$$

Integrals over $\mathbf{R}_a^n$

The steps of the above proof illustrate how multiple integrals are built up by composition of single integrals. The general theory of integrals over $\mathbf{R}_a^n (\overset{\text{def}}{=} \mathbf{R}_a \times \ldots \times \mathbf{R}_a$, n factors) is based on such composition. The 'volume element' in $\mathbf{R}_a^n$ is defined to be

$$\mathrm{d}^n x \overset{\text{def}}{=} i^{n(n-1)/2}\,\mathrm{d}x^1 \ldots \mathrm{d}x^n, \tag{1.3.19}$$

where $(x^1, \ldots, x^n)$ denotes an arbitrary point of $\mathbf{R}_a^n$. The n-dimensional delta function is then given by

$$\delta(x) = (2\pi\mathrm{i})^{-n/2}\mathrm{i}^{n(n-1)/2}x^1 \ldots x^n, \tag{1.3.20}$$

where 'x' is an abbreviation for '$(x^1, \ldots, x^n)$'. The proof is straightforward:

$$\int f(x)\delta(x)\,\mathrm{d}^n x = (2\pi\mathrm{i})^{-n/2}(-1)^{n(n-1)/2}\int f(x)x^1 \ldots x^n\,\mathrm{d}x^1 \ldots \mathrm{d}x^n$$

$$= (2\pi\mathrm{i})^{-n/2} f(0)\int x^1 \ldots x^n\,\mathrm{d}x^n \ldots \mathrm{d}x^1$$

$$= (2\pi\mathrm{i})^{-n/2} f(0)\int x^1\,\mathrm{d}x^1 \ldots \int x^n\,\mathrm{d}x^n = f(0).$$

Here $f(x)$ is any differentiable function on $\mathbf{R}_a^n$, namely any function having the general form

$$f(x) = \sum_{r=0}^{n}\frac{1}{r!}a_{\alpha_1 \ldots a_r}x^{a_r} \ldots x^{\alpha_1}, \tag{1.3.21}$$

where the indices α_i range from 1 to n and the a's are arbitrary supernumbers completely antisymmetric in these indices.

The delta function may also be expressed as a Fourier integral:

$$\delta(x) = (2\pi)^{-n}\int \mathrm{e}^{\mathrm{i}p_\alpha x^\alpha}\,\mathrm{d}^n p$$

$$= (2\pi)^{-n}\mathrm{i}^{n(n-1)/2}\int \mathrm{e}^{\mathrm{i}p_1 x^1} \ldots \mathrm{e}^{\mathrm{i}p_n x^n}\,\mathrm{d}p_1 \ldots \mathrm{d}p_n$$

$$= (2\pi)^{-n} i^{n(n-1)/2} \int e^{ip_1 x^1} dp_1 \ldots e^{ip_n x^n} dp_n$$
$$= i^{n(n-1)/2} \delta(x^1) \ldots \delta(x^n). \tag{1.3.22}$$

An alternative derivation is the following:

$$(2\pi)^{-n} \int e^{ip_\alpha x^\alpha} d^n p = (2\pi)^{-n} i^{n(n-1)/2} \int \sum_{r=0}^{n} \frac{(-1)^r}{r!} x^{\alpha_1} p_{\alpha_1} \ldots x^{\alpha_r} p_{\alpha_r} dp_1 \ldots dp_n$$
$$= (2\pi i)^{-n} i^{n(n-1)/2} \frac{1}{n!} \int x^{\alpha_1} \ldots x^{\alpha_n} p_{\alpha_n} \ldots p_{\alpha_1} dp_1 \ldots dp_n$$
$$= (2\pi i)^{-n} i^{n(n-1)/2} x^1 \ldots x^n \int p_n \ldots p_1 dp_1 \ldots dp_n$$
$$= (2\pi i)^{-n/2} i^{n(n-1)/2} x^1 \ldots x^n.$$

The n-dimensional Fourier transform is defined by

$$\tilde{f}(p) = (2\pi)^{-n/2} \int f(x) e^{ix^\alpha p_\alpha} d^n x. \tag{1.3.23}$$

When $f(x)$ is expressed in the general form (1.3.21) its Fourier transform takes the form

$$\tilde{f}(p) = \sum_{r=0}^{n} \frac{1}{r!} \tilde{a}^{\alpha_1 \ldots \alpha_r} p_{\alpha_r} \ldots p_{\alpha_1}, \tag{1.3.24}$$

where

$$\tilde{a}^{\alpha_1 \ldots \alpha_r} = i^{n/2} (-i)^{n(n-1)/2} \frac{(-1)^{nr} i^r}{(n-r)!} a_{\beta_1 \ldots \beta_{n-r}} \varepsilon^{\beta_{n-r} \ldots \beta_1 \alpha_1 \ldots \alpha_r}, \tag{1.3.25}$$

ε being the antisymmetric permutation symbol with n indices. Using the fact that

$$\delta(-x) = (-1)^n \delta(x), \tag{1.3.26}$$

where '$-x$' is an abbreviation for '$(-x^1, \ldots, -x^n)$', one may show that $\tilde{\tilde{f}} = f$ by a proof patterned on (1.3.18).

It should be obvious to the reader that integrals over $\mathbf{R}_a$ can be combined with integrals over $\mathbf{R}_c$ to produce multiple integrals over $\mathbf{R}_c^m \times \mathbf{R}_a^n$. Such integrals play an important role in supermanifold theory and we shall devote considerable attention to them. One of the important problems is to determine the rules for changing variables in such integrals. Both linear and nonlinear transformations of variables will be studied. But before undertaking this task we shall find it convenient to introduce the concept of a *supervector space*.