# Phys 503: Midterm Exam 1

**Fall 2011**

- Write your name and Student ID number in the space provided below and sign.

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| **Name, Last Name:** | |
| **ID Number:** | |
| **Signature:** | |

- You have 80 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

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| **Estimated Grade:** | |

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

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## To be filled by the grader:

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| **Actual Grade:** | |
| **Adjusted Grade:** | |

**Problem 1.** Consider a classical point particle of mass $m$ that moves on a straight line under the influence of a conservative force $F = F(x)$ where $x$ denotes the position of the particle.

**1.a)** Prove that the total energy of the system is conserved and explain how one can use this fact to integrate the equation of motion (10 points)

**1.b)** Let $(x, p)$ be the standard position and momentum observables for this system and $y := ax + bp$ and $q := cx + dp$ where $a, b, c, d$ are real numbers. Find the necessary and sufficient condition on these numbers such that $(x, p) \to (y, q)$ be a canonical transformation. (10 points)

**Problem 2.** Explain the difference between the following pair of concepts. (8 points)

**2.a)** inner product space and Hilbert space:

**2.b)** dense subspace and closed subspace of an inner product space:

**2.c)** symmetric operator and self-adjoint operator:

**2.d)** isometry and unitary operator:

**Problem 3.** Show that if $\langle\cdot|\cdot\rangle : \mathbb{C} \to \mathbb{C}$ is an inner product on $\mathbb{C}$, then it must have the following form for some positive real number $r$: For all $w, z \in \mathbb{C}$, $\langle w|z\rangle = rw^*z$. (12 points)

**Problem 4.** Let $\mathbb{E}^2$ be the Hilbert space obtained by endowing $\mathbb{C}^2$ with the Euclidean inner product, $\phi$ be a real number, $b_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $b_2 := \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$, and $\hat{U} : \mathbb{E}^2 \to \mathbb{E}^2$ be a linear operator defined on $\mathbb{E}^2$ that satisfies $\hat{U}b_1 = e^{i\phi}b_2$ and $U\hat{b}_2 = e^{-i\phi}b_1$.

**4.a)** Show that $(e^{i\phi}b_2, e^{-i\phi}b_1)$ is an orthonormal basis of $\mathbb{E}^2$ for all $\phi \in \mathbb{R}$. (10 points)

**4.b)** Show that $\hat{U}$ is a unitary operator. (10 points)

**Problem 5.** Let $\mathbb{E}^2$ be the Hilbert space obtained by endowing $\mathbb{C}^2$ with the Euclidean inner product and $b_1$ and $b_2$ be the vectors defined in Problem 2 and $a := b_1 - b_2$.

**5.a)** Find the matrix representation of the projection operator $\hat{P}_a$ onto the ray defined by the state vector $a$ in the standard basis of $\mathbb{E}^2$. (10 points)

**5.b)** Let $\lambda$ be the state defined by the state vector $\psi_\lambda := 2b_1 + 3b_2$. Find the probability of finding 1 upon measuring $\hat{P}_a$ in the state $\lambda$. (15 points)

**5.c)** Calculate the expectation value of measuring $\hat{O} := |a\rangle\langle a| - 2|b_1\rangle\langle b_1| - |b_2\rangle\langle b_2|$ in the state $\lambda$. (15 points)