Solutions

Phys 501: Final Exam

Fall 2014

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.

Problem 1 Consider a particle moving on the real line. Suppose that the dynamics of this particle is determined by the Hamiltonian

$$H = \frac{q^4 p^2}{2\mu} + \frac{\lambda}{q^2},$$

where μ and λ are positive real constants.

1.a (5 points) Write down the Hamilton's equations of motion for this system and simplify them as much as possible. Do not try to solve them.

1.b (10 points) Find a Lagrangian for this system and write down the corresponding Euler-Lagrange equation. You are expected to simplify this equation as much as possible, but not solve it.

Problem 2 Let (q, p) denote the usual coordinate and momentum pair in the phase space $\mathbb{R}^2$ of a classical system. Consider the transformation $(q, p) \to (\tilde{q}, \tilde{p})$ given by

$$\tilde{q} := \alpha\, q^a p, \qquad \tilde{p} := \beta\, q^b,$$

where α, β, a and b are real parameters.

2.a (10 points) Find a and b such that $(q, p) \to (\tilde{q}, \tilde{p})$ is a canonical transformation.

2.b (5 points) Perform ~~the inverse of~~ this canonical transformation on the system given in Problem 1 and find the transformed Hamiltonian.

2.c (15 points) Use your response to Part b of this problem to obtain the solution of the equations of motion for the system of Problem 1, i.e., give explicit formulas for $q(t)$ and $p(t)$.

Problem 3 For a solid hemisphere of radius a and mass m compute the following quantities.

3.a (5 points) Center of mass;

3.b (15 points) Principal axes and principal moments of inertia.

Problem 4 The dynamics of a system with one degree of freedom is determined by the Hamiltonian $H := \alpha e^{-q}p + \beta e^{-2q}$ where $(q, p) \in \mathbb{R}^2$ and α and β are positive real parameters. Suppose that at $t = 0$ the system is in the state given by $(q, p) = (0, 0)$.

4.a (10 points) Write down the the Hamilton-Jacobi equation for this system and find a complete solution for this equation.

4.b (10 points) Use your response to part a of this problem to determine $q(t)$ and $p(t)$ for $t > 0$.

Problem 5 Let $\vec{r} := (x, y)$ be the position of a particle that moves in a plane and has the potential energy given by $V(\vec{r}) = -\frac{\alpha}{r}$ where $r := |\vec{r}|$ and α is a real parameter. Let $\vec{p} := (p_x, p_y)$ denote the momentum of the particle and $A := p_y + \frac{\beta x}{r}$, where β is a real constant.

5.a (10 points) Write down the Hamiltonian of the system and compute its Poisson bracket with A.

5.b (5 points) Is there any value of β for which A is a constant of motion? Provide a justification for your response.

Problem 1: $H = \frac{q^4 p^2}{2\mu} + \frac{\lambda}{q^2}$

a) $\dot{q} = \frac{\partial H}{\partial p} = \frac{q^4 p}{\mu}$

$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{2}{\mu} q^3 p^2 + \frac{2\lambda}{q^3} = -2\left(\frac{q^3 p^2}{\mu} - \frac{\lambda}{q^3}\right)$

b) $p = \frac{\mu \dot{q}}{q^4}$ $\quad H = \dot{q}p - L \Rightarrow L = \dot{q}p - H$

$\Rightarrow L = \frac{\mu \dot{q}^2}{q^4} - \underbrace{\left(\frac{q^4}{2\mu}\right)\left(\frac{\mu \dot{q}}{q^4}\right)^2}_{\frac{\mu \dot{q}}{2q^4}} - \frac{\lambda}{q^2}$

$\Rightarrow \boxed{L = \frac{\mu \dot{q}^2}{2q^4} - \frac{\lambda}{q^2}}$

$\frac{\partial L}{\partial \dot{q}} = \frac{\mu \dot{q}}{q^4}$, $\frac{\partial L}{\partial q} = -\frac{2\mu \dot{q}^2}{q^5} + \frac{2\lambda}{q^3}$

$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \Rightarrow \mu \underbrace{\frac{d}{dt}\frac{\dot{q}}{q^4}}_{\frac{\ddot{q}}{q^4} - \frac{4\dot{q}^2}{q^5}} = -\frac{2\mu \dot{q}^2}{q^5} + \frac{2\lambda}{q^3}$

$\Rightarrow \frac{\ddot{q}}{q^4} - \frac{4\dot{q}^2}{q^5} = -\frac{2\dot{q}^2}{q^5} + \frac{2\lambda}{\mu q^3}$

$\Rightarrow \boxed{\ddot{q} - \frac{2\dot{q}^2}{q} - \frac{2\lambda}{\mu} q = 0}$

Problem 2 : a) $\{\tilde{q}, \tilde{p}\} = 1$

$$\Leftrightarrow \quad \underbrace{\frac{\partial \tilde{q}}{\partial q}}_{\alpha a q^{a-1} p} \underbrace{\frac{\partial \tilde{p}}{\partial p}}_{0} - \underbrace{\frac{\partial \tilde{q}}{\partial p}}_{\alpha q^{a}} \underbrace{\frac{\partial \tilde{p}}{\partial q}}_{\beta b q^{b-1}} = 1$$

$$\Leftrightarrow \quad -\alpha \beta b \, q^{a+b-1} = 1 \quad \Leftrightarrow \quad \begin{cases} a+b-1 = 0 \\ -\alpha\beta b = 1 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} a = 1 + \dfrac{1}{\alpha\beta} & (1) \\ b = -\dfrac{1}{\alpha\beta} & (2) \end{cases}$$

2.b) Inverse tramformation : $\tilde{q} = \alpha q^{a} p \, , \quad \tilde{p} = \beta q^{b}$

$$H = \frac{1}{2\mu} (q^2 p)^2 + \lambda \left(\frac{1}{q^2} \right)$$

Let $\boxed{a = 2} \Rightarrow \alpha\beta = 1 \overset{(1)}{\underset{(2)}{\Rightarrow}} \boxed{\begin{cases} b = -1 \\ \beta = \dfrac{1}{\alpha} \end{cases}}$

$(q, p) \to (\tilde{q}, \tilde{p})$ is a time-indep. C.T $\Rightarrow$

$$H(q,p) \longrightarrow K(\tilde{q}, \tilde{p}) = H(q(\tilde{q}, \tilde{p}), p(\tilde{q}, \tilde{p}))$$

$$\Rightarrow \quad K = \lambda \alpha^2 \tilde{p}^2 + \frac{1}{2\mu\alpha^2} \tilde{q}^2 = \lambda\alpha^2 \left(\tilde{p}^2 + \frac{1}{2\mu\lambda\alpha^4} \tilde{q}^2 \right)$$

α is still arbitrary.

Take $\alpha := (2\mu\lambda)^{-\frac{1}{4}} \Rightarrow \boxed{K = \sqrt{\frac{\lambda}{2\mu}} (\tilde{p}^2 + \tilde{q}^2)}$

2.c)

$$\dot{\tilde{q}} = \frac{\partial K}{\partial \tilde{p}} = \sqrt{\frac{2\lambda}{\mu}}\, \tilde{p} \qquad \Gamma \qquad \ddot{\tilde{q}} = -\frac{2\lambda}{\mu}\tilde{q}$$

$$\dot{\tilde{p}} = -\frac{\partial K}{\partial \tilde{q}} = -\sqrt{\frac{2\lambda}{\mu}}\, \tilde{q} \qquad\qquad \Downarrow$$

$$\ddot{\tilde{q}} + \frac{2\lambda}{\mu}\tilde{q} = 0$$

$$\Rightarrow \begin{cases} \tilde{q}(t) = A \sin\left[\sqrt{\frac{2\lambda}{\mu}}\,(t-t_0)\right] \\ \tilde{p}(t) = \sqrt{\frac{\mu}{2\lambda}}\, \dot{q}(t) = A \cos\left[\sqrt{\frac{2\lambda}{\mu}}\,(t-t_0)\right] \end{cases}$$

$$\tilde{p} = \beta q^{-1} = \frac{1}{\alpha} q^{-1} \Rightarrow \quad q = \frac{1}{\alpha \tilde{p}} = \frac{(2\mu\lambda)^{\frac{1}{4}}}{\tilde{p}}$$

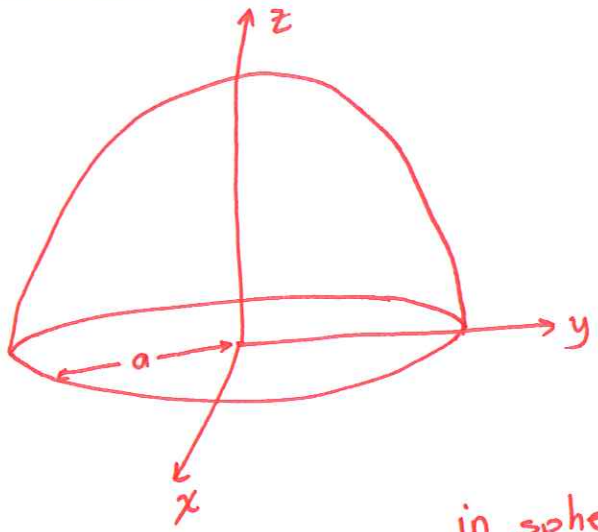
$$\tilde{q} = \alpha q^2 p \Rightarrow \quad p = \frac{1}{\alpha} \frac{\tilde{q}}{\left(\frac{1}{\alpha \tilde{p}}\right)^2} = \alpha \tilde{q} \tilde{p}^2 = (2\mu\lambda)^{-\frac{1}{4}} \tilde{q} \tilde{p}^2$$

$$\Rightarrow \quad q(t) = \frac{(2\mu\lambda)^{\frac{1}{4}}}{A \cos\left[\sqrt{\frac{2\lambda}{\mu}}\,(t-t_0)\right]}$$

$$p(t) = (2\mu\lambda)^{-\frac{1}{4}} \frac{\sin\left[\sqrt{\frac{2\lambda}{\mu}}\,(t-t_0)\right]}{A \cos^2\left[\sqrt{\frac{2\lambda}{\mu}}\,(t-t_0)\right]}$$

$$\text{let} \begin{cases} q_0 := \frac{(2\mu\lambda)^{\frac{1}{4}}}{A} \\ \omega := \sqrt{\frac{2\lambda}{\mu}} \end{cases} \Rightarrow \quad \boxed{\begin{aligned} q(t) &= \frac{q_0}{\cos[\omega(t-t_0)]} \\ p(t) &= \frac{1}{q_0\sqrt{2\mu\lambda}} \frac{\sin[\omega(t-t_0)]}{\cos[\omega(t-t_0)]} \end{aligned}}$$

Problem 3:



mass density $= \rho = \frac{m}{V} = \frac{m}{\frac{2}{3}\pi a^3}$

$\Rightarrow \rho = \frac{3m}{2\pi a^3}$

a) Due to symmetry, $x_{cm} = y_{cm} = 0$

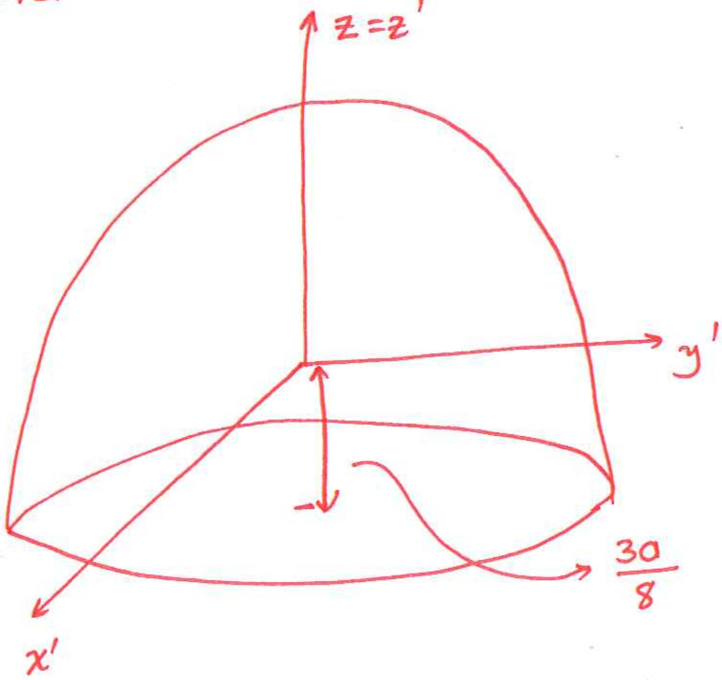
$$z_{cm} = \frac{1}{M} \int_V \rho z \, dV$$

in spherical coordinates

$z = r\cos\theta \qquad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$\Rightarrow z_{cm} = \frac{\rho}{m} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta \, d\theta \int_{r=0}^{a} r^3 dr = \left(\frac{3}{2\pi a^3}\right)(2\pi)\left(\frac{1}{2}\right)\left(\frac{a^4}{4}\right) = \frac{3a}{8}$$

b) Inertia tensor w.r.t. axes passing through C.M.:



By symmetry,

$I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = 0$

Thus, axes shown are the principal axes.

Also, by symmetry $I_{11} = I_{22}$

$I_{11} = J_{11} - m\left(\frac{3a}{8}\right)^2$, $\quad J_{11}$ = moment of inertia w.r.t. original axes.

$$J_{11} = \rho \int_V (y^2+z^2)\, dV = \rho \int (r^2\sin^2\theta \sin^2\phi + r^2\cos^2\theta)\, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{3m}{2\pi a^3} \int_0^a r^4 dr \int_{\theta=0}^{\pi/2} \left[\int_{\phi=0}^{2\pi} (\sin^2\theta \sin^2\phi + \cos^2\theta)\, d\phi\right] \sin\theta \, d\theta$$

$$= \frac{3ma^2}{10\pi} \int_{\theta=0}^{\pi/2} (\pi \sin^3\theta + 2\pi \cos^2\theta \sin\theta)\, d\theta = \frac{2}{5} M a^2$$

$$\Rightarrow \quad I_{11} = I_{22} = \frac{2}{5} ma^2 - \frac{9}{64} ma^2 = \frac{83}{320} ma^2$$

Also, $I_{33} = J_{33} - M \cdot 0 = J_{33}$

$$I_{33} = \rho \int_V (x^2+y^2)\, dV$$

$$= \rho \int r^4 \sin^3\theta \, dr\, d\theta\, d\phi = \frac{2}{5} ma^2$$

Thus, the principal axes are the primed axes shown in the figure. The principal moments of inertia are

$$I_{11} = I_{22} = \frac{83}{320} ma^2 \qquad I_{33} = \frac{2}{5} ma^2$$

Problem 4 : a) $H = \alpha e^{-q} p + \beta e^{-2q}$

$$-\frac{\partial S}{\partial t} = H\left(q, \frac{\partial S}{\partial q}\right) , \quad \frac{\partial S}{\partial q} = p , \quad \frac{\partial S}{\partial \tilde{q}} = -\tilde{p}$$

$$S = -\underset{\substack{\downarrow \\ \tilde{q}}}{E}t + W(q, E) \Rightarrow \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} =: W'$$

$$E = \alpha e^{-q} W' + \beta e^{-2q}$$

$$\Rightarrow W' = \frac{1}{\alpha} e^{q} (E - \beta e^{-2q}) = \frac{E}{\alpha} e^{q} - \frac{\beta}{\alpha} e^{-q}$$

$$\Rightarrow W = \frac{E}{\alpha} e^{q} + \frac{\beta}{\alpha} e^{-q} + W_0$$

a constant that we can ignore

$$\frac{\partial^2 S}{\partial q \partial \tilde{q}} = \frac{\partial^2 W}{\partial q \partial E} = \frac{e^{q}}{\alpha} \neq 0 \Rightarrow$$

$$\boxed{S = -Et + \frac{E}{\alpha} e^{q} + \frac{\beta}{\alpha} e^{-q}}$$

b) $$\frac{\partial S}{\partial \tilde{q}} = \frac{\partial S}{\partial E} = -t + \frac{e^{q}}{\alpha} = -\tilde{p}$$

$$\Rightarrow e^{q} = \alpha(t - \tilde{p}) \Rightarrow q(t) = \ln[\alpha(t - \tilde{p})]$$

$$q(0) = 0 \Rightarrow -\alpha \tilde{p} = 1$$

$$\tilde{p} = -\frac{1}{\alpha}$$

$$\Rightarrow \boxed{q(t) = \ln(\alpha t + 1)}$$

$$p = W' = \frac{E}{\alpha} e^{q} - \frac{\beta}{\alpha} e^{-q} = \frac{E}{\alpha}(\alpha t + 1) - \frac{\beta}{\alpha}(\alpha t + 1)^{-1}$$

$$p(0) = 0 \Leftrightarrow \frac{E}{\alpha} - \frac{\beta}{\alpha} = 0 \Rightarrow E = \beta$$

$$\Rightarrow \boxed{p(t) = \frac{\beta}{\alpha}\left(\alpha t + 1 - \frac{1}{\alpha t + 1}\right)}$$

Problem 5 : a) $H = \frac{\vec{p}^{\,2}}{2m} - \frac{\alpha}{r}$, $A = P_y + \frac{\beta x}{r}$

$$r = (x^2+y^2)^{1/2}$$

$$\frac{\partial H}{\partial x} = -\alpha \frac{\partial r^{-1}}{\partial x} = +\alpha \frac{x}{r^3} , \quad \frac{\partial H}{\partial y} = \alpha \frac{y}{r^3}$$

$$\frac{\partial H}{\partial p_x} = \frac{P_x}{m} , \quad \frac{\partial H}{\partial P_y} = \frac{P_y}{m}$$

$$\frac{\partial A}{\partial x} = \beta\left(\frac{1}{r} + x\frac{\partial r^{-1}}{\partial x}\right) = \beta\left(\frac{1}{r} - \frac{x^2}{r^3}\right) = \frac{\beta y^2}{r^3}$$

$$\frac{\partial A}{\partial y} = -\frac{\beta x y}{r^3} , \quad \frac{\partial A}{\partial P_x} = 0 , \quad \frac{\partial A}{\partial P_y} = 1$$

$$\{A,H\} = \frac{\partial A}{\partial x}\frac{\partial H}{\partial p_x} + \frac{\partial A}{\partial y}\frac{\partial H}{\partial P_y} - \overbrace{\frac{\partial A}{\partial P_x}}^{0}\frac{\partial H}{\partial x} - \frac{\partial A}{\partial P_y}\frac{\partial H}{\partial y}$$

$$= \left(\frac{\beta y^2}{r^3}\right)\left(\frac{P_x}{m}\right) - \left(\frac{\beta x y}{r^3}\right)\left(\frac{P_y}{m}\right) - \frac{\alpha y}{r^3}$$

$$= -\frac{\alpha y}{r^3}\left[1 - \frac{\beta}{\alpha m}(y P_x - x P_y)\right]$$

$$= \frac{\alpha y}{r^3}\left[\frac{\beta}{\alpha m}(x P_y - y P_x) - 1\right]$$

b) Because $\frac{1}{r}$ is a central potential, angular momentum $\vec{L} = (x P_y - y P_x)\hat{k}$ is conserved

i.e, $x P_y - y P_x = \text{const} = l$.

For $\beta = \frac{\alpha m}{l}$ we have $\{A,H\} = 0$

Therefore A is a constant of motion.