Phys 501: Midterm Exam 2

Fall 2014

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours & 45 minutes.
- You can earn 10 implicit bonus points, for the maximum possible grade is 110 out of 100.

Problem 1 Consider three particles of mass m that are confined to move on a circle of radius a centered at the origin of a polar coordinate system (r, φ) in a plane. Let us parameterize the position of these article by their polar angle φ_j with $j = 1, 2, 3$. Each of these particles interacts only with its two neighboring particles via a force that is a linear function of the arc length separating them. More specifically the potential energy due to the force between particles i and j is given by $\frac{ka^2}{2}(\varphi_i - \varphi_j)^2$, where k is a positive real parameter.

1.a (5 points) Write down a Lagrangian for this system (using φ_j as the dynamical variables.)

$$L = \frac{m}{2} a^2(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2) - \frac{ka^2}{2}\left[(\varphi_1-\varphi_2)^2 + (\varphi_2-\varphi_3)^2 + (\varphi_1-\varphi_3)^2\right]$$

1.b (5 points) Determine equations of motion for this system and express them in the form $\ddot\Phi(t) + \Lambda\Phi(t) = 0$, where $\Phi(t) := \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix}$ and Λ is a 3×3 matrix that you need to find.

$$\frac{\partial L}{\partial \dot\varphi_i} = m a^2 \dot\varphi_i \quad , \quad i = 1, 2, 3$$

$$\frac{\partial L}{\partial \varphi_1} = -ka^2\left[(\varphi_1-\varphi_2) + (\varphi_1-\varphi_3)\right] = -ka^2(2\varphi_1 - \varphi_2 - \varphi_3)$$

$$\frac{\partial L}{\partial \varphi_2} = -ka^2(2\varphi_2 - \varphi_1 - \varphi_3)$$

$$\frac{\partial L}{\partial \varphi_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi}$$

$$\frac{\partial L}{\partial \varphi_3} = -ka^2(2\varphi_3 - \varphi_1 - \varphi_2)$$

$$\text{let } \omega_o := \sqrt{\frac{k'}{m}} \Rightarrow$$

$$\begin{cases} \ddot\varphi_1 = -\omega_o^2(2\varphi_1 - \varphi_2 - \varphi_3) \\ \ddot\varphi_2 = -\omega_o^2(2\varphi_2 - \varphi_1 - \varphi_3) \\ \ddot\varphi_3 = -\omega_o^2(2\varphi_3 - \varphi_1 - \varphi_2) \end{cases} \Rightarrow \ddot{\vec\Phi} + \Lambda\vec\Phi = 0 \text{ for } \Lambda = \omega_o^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

1.c (10 points) Find the normal modes of this system and the corresponding frequencies.

$$(\Lambda - \omega^2 I)\vec{a} = \vec{0} \Rightarrow \underbrace{\left(\frac{1}{\omega_0^2}\Lambda - \underbrace{\left(\frac{\omega}{\omega_0}\right)^2}_{\lambda} I\right)\vec{a}}_{} = \vec{0}$$

$$\begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 = \det\begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^3 - 1 - 1 - [(2-\lambda)+(2-\lambda)+(2-\lambda)]$$

$$= -[(\lambda-2)^3 - 3(\lambda-2) + 2] = -[\lambda^3 - 6\lambda^2 + 12\lambda - 8 - 3\lambda + 6 + 2]$$

$$= -\lambda(\lambda^2 - 6\lambda + 9) = -\lambda(\lambda-3)^2 \Rightarrow \begin{cases} \lambda = 0 \\ \lambda = 3 \end{cases}$$

For $\lambda = 0$: $\boxed{\omega = \omega_1 = 0}$, $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{matrix} 2a_1 - a_2 - a_3 = 0 \\ -a_1 + 2a_2 - a_3 = 0 \end{matrix} \Rightarrow 3a_1 - 3a_2 = 0 \Rightarrow a_2 = a_1$$

$$a_3 = 2a_1 - a_2 = a_1$$

$$\Rightarrow \vec{a} = a_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{a}_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$: $\boxed{\omega = \omega_2 = \sqrt{3}\,\omega_0}$, $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow a_1 + a_2 + a_3 = 0 \qquad \Rightarrow \vec{a} = a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$a_3 = -(a_1 + a_2)$$

$$\Rightarrow \vec{a}_{2,1} := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{a}_{2,2} := \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

1.d (10 pt)

$$\Rightarrow \vec{\phi} = (a_0 t + b_0)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)e^{i\sqrt{3}\omega_0 t}$$

$$+ \left(b_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)e^{-i\sqrt{3}\omega_0 t}$$

Problem 2 Consider a particle of mass m moving in a plane whose dynamics is described by the Lagrangian: $L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{\alpha + \beta\dot{r}^2}{r}$, where (r, φ) are polar coordinates, and α and β are real parameters.

2.a (5 points) Find the expression for the momenta conjugate to r and φ, and determine the Hamiltonian for this system.

$$P_r := \frac{\partial L}{\partial \dot{r}} = m\dot{r} - \frac{2\beta}{r}\dot{r} = \left(m - \frac{2\beta}{r}\right)\dot{r}$$

$$\dot{r} = \frac{P_r}{m - \frac{2\beta}{r}}$$

$$P_\varphi := \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \Rightarrow \dot{\varphi} = \frac{P_\varphi}{m r^2}$$

$$\hat{L} = \frac{m}{2}\frac{P_r^2}{\left(m - \frac{2\beta}{r}\right)^2} - \frac{\alpha}{r} - \frac{\beta}{r}\frac{P_r^2}{\left(m - \frac{2\beta}{r}\right)^2} + \frac{P_\varphi^2}{2mr^2}$$

$$H = \dot{r}P_r + \dot{\varphi}P_\varphi - \hat{L}$$

$$= \frac{P_r^2}{\left(m - \frac{2\beta}{r}\right)} + \frac{P_\varphi^2}{m r^2} - \frac{m}{2}\frac{P_r^2}{\left(m - \frac{2\beta}{r}\right)^2} + \frac{\alpha}{r} + \frac{\beta}{r}\frac{P_r^2}{\left(m - \frac{2\beta}{r}\right)^2} - \frac{P_\phi^2}{2mr^2}$$

$$= \frac{P_r^2}{\left(m - \frac{2\beta}{r}\right)^2}\underbrace{\left(m - \frac{2\beta}{r} - \frac{m}{2} + \frac{\beta}{r}\right)}_{\frac{m}{2} - \frac{\beta}{r} = \frac{1}{2}\left(m - \frac{2\beta}{r}\right)} + \frac{P_\varphi^2}{m r^2} + \frac{\alpha}{r} - \frac{P_\varphi^2}{2mr^2}$$

$$\Rightarrow \boxed{H = \frac{P_r^2}{2\left(m - \frac{2\beta}{r}\right)} + \frac{P_\varphi^2}{2mr^2} + \frac{\alpha}{r}}$$

2.b (5 points) Use Hamilton's equations to show that the angular momentum of the particle is conserved.

angular moment $= P_\varphi$

$$\dot{P}_\varphi = \frac{\partial H}{\partial \varphi} = 0 \Rightarrow P_\varphi \text{ is conserved.}$$

2.c (10 points) Use the conservation of angular momentum and the value E of the Hamiltonian of the system to reduce the equations of motion to a single first order differential equation for $r(t)$.

$$E = \frac{P_r^2}{2\left(m - \frac{2\beta}{r}\right)} + \frac{P_\varphi^2}{2mr^2} + \frac{\alpha}{r}$$

$P_\varphi = \text{const} =: l$

$$P_r = \left(m - \frac{2\beta}{r}\right)\dot{r}$$

$$\Rightarrow \quad E = \frac{1}{2}\left(m - \frac{2\beta}{r}\right)\dot{r}^2 + \frac{l^2}{2mr^2} + \frac{\alpha}{r}$$

$$\Rightarrow \quad \dot{r}^2 = \left(\frac{2}{m - \frac{2\beta}{r}}\right)\left[E - \frac{l^2}{2mr^2} - \frac{\alpha}{r}\right]$$

$$\Rightarrow \quad \dot{r} = \sqrt{\frac{2\left(E - \frac{l^2}{2mr^2} - \frac{\alpha}{r}\right)}{m - \frac{2\beta}{r}}}$$

Sign can be absorbed in the definition of t.

2.d (5 points) Make the change of variables $t \to \varphi$ and $r \to u := 1/r$ to express the equation you find in part (2.c) as a first order differential equation for $u(\varphi)$.

$$\dot{\varphi} = \frac{\ell}{m r^2}, \quad u' = \frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\varphi}} = -\frac{m}{\ell}\dot{r}$$

$$u' = -\frac{m}{\ell}\sqrt{\frac{2\left(E - \frac{\ell^2}{2m}u^2 - \alpha u\right)}{m - 2\beta u}}$$

2.e (10 points) Solve the equation you find in part (2.d) to obtain φ as a function of u for the special case that $\alpha = 0$ and $E = 0$, and give an explicit formula for φ as a function of r. Hint: You may use the formula $\int u^{-1}\sqrt{\gamma u - 1}du = 2\left[\sqrt{\gamma u - 1} - \tan^{-1}(\sqrt{\gamma u - 1})\right]$.

$$\alpha = E = 0 \Rightarrow \quad u' = -\frac{m}{\ell}\sqrt{\frac{\frac{\ell^2}{m}u^2}{2\beta u - m}}$$

$$= -\frac{\sqrt{m}}{\sqrt{2}\sqrt{\beta}}\frac{u}{\sqrt{u - \frac{m}{2\beta}}}$$

$$\varphi = -\frac{\sqrt{2}\sqrt{\beta}}{\sqrt{m}}\int \sqrt{u - \frac{m}{2\beta}}\,\frac{du}{u}$$

$$= -\frac{2\sqrt{2}\sqrt{\beta}}{\sqrt{m}}\left[\sqrt{u - \frac{m}{2\beta}} - \sqrt{\frac{m}{2\beta}}\tan^{-1}\sqrt{\frac{u - m/2\beta}{\frac{m}{2\beta}}}\right]$$

$$= 2\left[-\sqrt{\frac{2\beta u}{m} - 1} + \tan^{-1}\sqrt{\frac{2\beta u}{m} - 1}\right]$$

$$= 2\left[\tan^{-1}\sqrt{\frac{2\beta}{mr} - 1} - \sqrt{\frac{2\beta}{mr} - 1}\right]$$

$$\frac{2\beta}{mr} - 1 \geq 0 \Rightarrow \quad mr \leq 2\beta \Rightarrow \quad \boxed{r \leq \frac{2\beta}{m}}$$

Problem 3 Consider a dynamical system with phase space $\mathbb{R}^{2n}$ and dynamical equations: $\dot{\xi}^i(t) = \sum_{j=1}^{2n} A^{ij}(t)\,\xi^j(t)$, where $i = 1, 2, \cdots, 2n$ and $A^{ij}(t)$ are entries of a $2n \times 2n$ real t-dependent matrix $\mathbf{A}(t)$.

3.a (10 points) Find a necessary and sufficient condition on $\mathbf{A}(t)$ such that this system is a Hamiltonian system.

$$\dot{\xi}^i = \omega^{ij} \frac{\partial H}{\partial \xi^j} \iff A^{ij} \xi^j = \omega^{ij} \frac{\partial H}{\partial \xi^j}$$

$$\omega^{ki} A^{ij} \xi^j = \omega^{ki} \omega^{ij} \frac{\partial H}{\partial \xi^j} = - \frac{\partial H}{\partial \xi^k}$$

$$\Rightarrow \frac{\partial^2 H}{\partial \xi^l \partial \xi^k} = \frac{\partial}{\partial \xi^l}(\omega^{ki} A^{ij} \xi^j) = \omega^{ki} A^{ij} \delta^j_l = \omega^{ki} A^{il}$$

$$\frac{\partial^2 H}{\partial \xi^k \partial \xi^l} \Rightarrow \omega^{ki} A^{il} = \omega^{li} A^{ik} = A^{T\,ki} \omega^{T\,il}$$

$$\omega \mathbb{A} = \mathbb{A}^T \omega^T = - \mathbb{A}^T \omega$$

$$\Downarrow$$

$$\mathbb{A}^T = - \omega \mathbb{A} \omega^{-1} = \omega \mathbb{A} \omega$$

3.b (10 points) Supposing that this condition is satisfied, find a Hamiltonian for this system

$$\frac{\partial H}{\partial \xi^k} = - \omega^{ki} A^{ij} \xi^j = - \sum_{i=1}^{2n} \sum_{j=1}^{2n} \omega^{ki} A^{ij} \xi^j$$

$$= - \sum_{i=1}^{2n} \sum_{\substack{j=1 \\ j \neq k}}^{2n} \omega^{ki} A^{ij} \xi^j - \sum_{i=1}^{2n} \omega^{ki} A^{ik} \xi^k$$

$$\Rightarrow H = - \sum_{k=1}^{2n} \left[\sum_{i=1}^{2n} \sum_{\substack{j=1 \\ j \neq k}}^{2n} \omega^{ki} A^{ij} \xi^j \xi^k + \sum_{i=1}^{2n} \omega^{ki} A^{ik} \frac{(\xi^k)^2}{2} \right]$$

Problem 4 Recall that a $2n \times 2n$ real matrix M is said to be symplectic if $M\omega M^T = \omega$, where ω is the standard symplectic matrix.

4.a (5 points) Show that every symplectic matrix is invertible. $\omega^2 = -\mathbb{I} \Rightarrow \det \omega \neq 0$

$\Rightarrow \det M\omega M^T \neq 0 \Rightarrow \det(M)\det(\omega)\det(M^T) \neq 0$

$\Rightarrow (\det M)^2 \det(\omega) \neq 0 \Rightarrow \det M \neq 0 \Rightarrow M$ is invertible

4.b (5 points) Show that product of every pair of symplectic matrices is symplectic.

Let M_1 & M_2 be symplectic: $M_1\omega M_1^T = \omega$, $M_2\omega M_2^T = \omega$

$M_1M_2\,\omega(M_1M_2)^T = M_1M_2\,\omega\, M_2^T M_1^T = M_1\omega M_1^T = \omega$. ☑

4.c (5 points) Show that the inverse of every symplectic matrix is symplectic.

If M is symplectic, $M\omega M^T = \omega$ ① $\Rightarrow M\omega = \omega(M^T)^{-1}$

$M^{-1}\omega(M^{-1})^T = M^{-1}\omega(M^T)^{-1} = (M^T\omega^{-1}M)^{-1}$ ②

$M^T\omega^{-1}M = -M^T\omega M = +M^T\omega \underbrace{M\omega}_{\omega(M^T)^{-1}}\omega = -M^T(M^T)^{-1}\omega$

$= -\omega$ ③

③ & ② $\Rightarrow$ $M^{-1}\omega(M^{-1})^T = (-\omega)^{-1} = -\omega^{-1} = \omega$ ☑

4.d (5 points) Show that if $\xi^i \to \eta^i(\xi^1, \xi^2, \cdots, \xi^{2n}, t)$ is a local canonical transformation, then the Jacobian of this transformation is a symplectic matrix.

$\{\eta^i, \eta^j\} = \omega^{ij}$ $\qquad \eta^i \frac{\overleftarrow{\partial}}{\partial \xi^k}\omega^{kl}\frac{\overrightarrow{\partial}}{\partial \xi^l}\eta^j = \omega^{ij}$

$\Rightarrow \underbrace{\frac{\partial \eta^i}{\partial \xi^k}}_{J^{ik}}\omega^{kl}\underbrace{\frac{\partial \eta^j}{\partial \xi^l}}_{J^{jl} = (J^T)^{lj}} = \omega^{ij}$

$\Rightarrow J^{ik}\omega^{kl}(J^T)^{lj} = \omega^{ij} \Rightarrow J\omega J^T = \omega$

$\Rightarrow J$ is symplectic.