Phys 501: Midterm Exam 1

Fall 2014

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2 hours and 45 minutes.

Problem 1 Consider a particle whose position is described by a single generalized coordinate q. Suppose that the motion of this particle is determined by the Lagrangian $L = \frac{m}{2}\left[\dot{q} - A(q)\right]^2 - \phi(q)$, where $A(q)$ and $\phi(q)$ are given smooth functions of q.

1.a (5 points) Find and simplify the Euler-Lagrange equation for this system.

$$\frac{\partial L}{\partial q} = -m(\dot{q} - A)A' - \phi' \quad , \quad \frac{\partial L}{\partial \dot{q}} = m(\dot{q} - A)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\Rightarrow \quad m(\ddot{q} - A'\dot{q}) - \left[-m\dot{q}A' + mAA' - \phi'\right] = 0$$

$$\Rightarrow \quad \boxed{m\ddot{q} - mAA' + \phi' = 0}$$

1.b (15 points) Find the expression for the Jacobi operator $\hat{J}$ of this system. Recall that

$$\hat{J}\delta q(t) := \int_{t_0}^{t_1} d\tau \frac{\delta^2 S[q]}{\delta q(t)\delta q(\tau)} \delta q(\tau).$$

$$\frac{\delta S[q]}{\delta q(\tau)} = \left(\frac{\partial L}{\partial q} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot q}\right)\Big|_{\tau}$$

$$\frac{\delta^2 S}{\delta q(t)\,\delta q(\tau)} = \lim_{\epsilon\to 0}\Big\{ \frac{\partial L}{\partial q}\big(q(\tau)+\epsilon\,\delta_t(\tau),\ \dot q(\tau)+\epsilon \tfrac{d}{d\tau}\delta_t(\tau)\big)$$

$$- \frac{d}{d\tau}\frac{\partial L}{\partial \dot q}\big(q(\tau)+\epsilon\,\delta_t(\tau),\ \dot q(\tau)+\epsilon\tfrac{d}{d\tau}\delta_t(\tau)\big)\Big\}\Big|_{\epsilon=0}$$

$$= \frac{\partial^2 L}{\partial q^2}\delta(t-\tau) + \frac{\partial^2 L}{\partial \dot q\partial q}\frac{d}{d\tau}\delta(t-\tau)$$

$$- \frac{d}{d\tau}\left[\frac{\partial^2 L}{\partial q\partial \dot q}\delta(t-\tau) + \frac{\partial^2 L}{\partial \dot q^2}\frac{d}{d\tau}\delta(t-\tau)\right]$$

$$= \frac{\partial^2 L}{\partial q^2}\delta(t-\tau) - \frac{d}{d\tau}\left(\frac{\partial^2 L}{\partial q\partial \dot q}\right)\delta(t-\tau) - \frac{d}{d\tau}\left[\frac{\partial^2 L}{\partial \dot q^2}\frac{d}{d\tau}\delta(t-\tau)\right]$$

$$\Rightarrow \hat{J}\delta q(t) = \left[\frac{\partial^2 L}{\partial q^2} - \frac{d}{d\tau}\frac{\partial^2 L}{\partial q\partial \dot q}\right]\delta q(t) - \int_{t_0}^{t_1} d\tau \frac{d}{d\tau}\left[\frac{\partial^2 L}{\partial \dot q^2}\frac{d}{d\tau}\delta(t-\tau)\right]\delta q(\tau)$$

$$= \left[\frac{\partial^2 L}{\partial q^2} - \frac{d}{d\tau}\frac{\partial^2 L}{\partial q\partial \dot q}\right]\delta q(t) + \int_{t_0}^{t_1} d\tau \frac{\partial^2 L}{\partial \dot q^2}\frac{d}{d\tau}\delta(t-\tau)\frac{d}{d\tau}\delta q(\tau)$$

$$- \underbrace{\left(\frac{\partial^2 L}{\partial \dot q^2}\frac{d}{d\tau}\delta(t-\tau)\,\delta q(\tau)\right)\Big|_{\tau=t_0}^{\tau=t_1}}_{\to 0}$$

$$= \left[\frac{\partial^2 L}{\partial q^2} - \frac{d}{d\tau}\frac{\partial^2 L}{\partial q\partial \dot q}\right]\delta q(t) - \frac{\partial^2 L}{\partial \dot q^2}\frac{d^2}{dt^2}\delta q(t) + \underbrace{\left[\frac{\partial^2 L}{\partial \dot q^2}\frac{d}{d\tau}\delta q(\tau)\,\delta(t-\tau)\right]\Big|_{\tau=t_0}^{\tau=t_1}}_{\to 0}$$

$$\Rightarrow \boxed{\hat{J} = -\frac{\partial^2 L}{\partial \dot q^2}\frac{d^2}{dt^2} + \left(\frac{\partial^2 L}{\partial q^2} - \frac{d}{d\tau}\frac{\partial^2 L}{\partial q\partial \dot q}\right)}$$

$$\frac{\partial^2 L}{\partial \dot q^2} = m, \quad \frac{\partial^2 L}{\partial q\partial \dot q} = -mA', \quad \frac{\partial^2 L}{\partial q^2} = -m(-A'^2 - AA'') - \phi'' = m(A'^2 + AA'') - \phi''$$

$$\Rightarrow \boxed{\hat{J} = -m\frac{d^2}{dt^2} + [m(A'^2 + AA'') - \phi'' + m A''\dot{q}]}$$

Problem 2 A particle of mass m is attached to one end of a rigid massless rod of length ℓ. The other end of the rod is pinned to the point $(a, 0, 0)$ in a Cartesian coordinate system, where a is a positive real parameter. The the particle is constrained to move on the cylinder defined by $x^2 + y^2 = a^2$ under the influence of a constant gravitational force that is along the negative z-axis.

2.a (10 points) Use cylindrical coordinates (r, θ, z) to write down a Lagrangian for this system that involves two Lagrange multipliers.

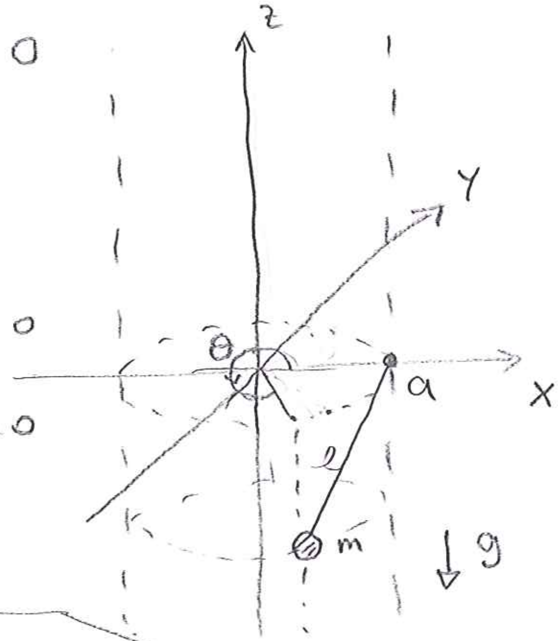
Constraints: $r = a \Rightarrow \phi_1 := r - a = 0$

$(x-a)^2 + y^2 + z^2 = \ell^2$

$\Rightarrow \phi_2 := (x-a)^2 + y^2 + z^2 - \ell^2 = 0$

$= x^2 - 2ax + a^2 + y^2 + z^2 - \ell^2 = 0$

$= r^2 - 2ar\cos\theta + a^2 + z^2 - \ell^2 = 0$



$$\boxed{L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz + \lambda_1(r-a) + \lambda_2(r^2 - 2ar\cos\theta + a^2 + z^2 - \ell^2)}$$

2.b (5 points) Impose the constraint $r = a$ and express the Lagrangian you found in part a of this problem in terms of the coordinates θ and z, and a single Lagrange multiplier.

$$r = a \Rightarrow \quad f_2 = z^2 - 2a^2 \cos\theta + 2a^2 - \ell^2 = 0 \quad \& \quad \dot{r} = 0$$

$$L = \frac{m}{2}\left(a^2\dot{\theta}^2 + \dot{z}^2\right) - mgz + \underset{\substack{\parallel \\ \lambda_2}}{\lambda}\left(z^2 - 2a^2\cos\theta + 2a^2 - \ell^2\right)$$

2.c (15 points) Derive and simplify the Euler-Lagrange equations corresponding to the Lagrangian you found in part b of this problem. Do not try to solve these equations.

$$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = 2\lambda a^2 \sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad ma^2\ddot{\theta} - 2\lambda a^2 \sin\theta = 0$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = \frac{2\lambda}{m}\sin\theta}$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}, \qquad \frac{\partial L}{\partial z} = -mg + 2\lambda z$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad m\ddot{z} + mg - 2\lambda z = 0$$

$$\Rightarrow \quad \boxed{\ddot{z} = -g + \frac{2\lambda}{m} z}$$

2.d (15 points) Find the values of θ and z that correspond to the equilibrium points of the system.
Hint: Consider the cases where $\ell \geq 2a$ and $\ell < 2a$ separately.

$$\phi_2 = 0 \Rightarrow \quad z^2 - 2a^2 \cos\theta + 2a^2 - \ell^2 = 0 \quad (*)$$

At equilibrium $\quad z = \text{const}, \quad \theta = \text{const}$

$$\Rightarrow \quad \frac{2\lambda}{m} \sin\theta = 0$$

$$-g + \frac{2\lambda}{m} z = 0 \Rightarrow \frac{2\lambda}{m} \neq 0 \Rightarrow \sin\theta = 0 \Downarrow \boxed{\theta = 0, \pi}$$

$$\Downarrow$$

$$z \neq 0 \quad (**)$$

- For $\theta = 0$, $\cos\theta = 1$ & $(*) \Rightarrow z^2 - \ell^2 = 0$

$$\Downarrow$$

$$\boxed{z = \pm \ell}$$

- For $\theta = \pi$, $\cos\theta = -1$ & $(*) \Rightarrow z^2 + 4a^2 - \ell^2 = 0$

$$\Downarrow$$

$$\boxed{z = \pm\sqrt{\ell^2 - 4a^2}}$$

$\Rightarrow \ell = 2a$ is not acceptable because of $(**)$

Therefore, if $\ell > 2a$, then are 4 equilibrium pts
namely: $(\theta, z) \in \{(0,\ell), (0,-\ell), (\pi, \sqrt{\ell^2 - 4a^2}), (\pi, -\sqrt{\ell^2-4a^2})\}$

if $\ell \leq 2a$, then are 2 equilibrium pts:
$(\theta, z) \in \{(0,\ell), (0,-\ell)\}$.

Problem 3 Consider a particle whose position is described by a single generalized coordinate q that takes values on the real line. Suppose that the motion of this particle is determined by a Lagrangian of the form $L(q,\dot q,t) = F(q,\dot q) + \alpha\, t\, \dot q$, where F is a smooth function of q and $\dot q$, and α is a real parameter.

3.a (**15** points) Use the transformation property of the action functional for this system under the time-translations $t \to t+\epsilon$ to determine a constant of motion C for this system.

$$t \to t+\epsilon \implies L \to \tilde L = F + \alpha t \dot q + \alpha \epsilon \dot q$$

$$\implies S[q] \to \tilde S[q] = \int_{t_0}^{t_1} \tilde L\, dt = S[q] + \alpha\epsilon \int_{t_0}^{t_1} \dot q\, dt$$

$$= S[q] + \alpha[q(t_1) - q(t_0)]\epsilon$$

$$\text{let } \phi(t) := \alpha q(t) \implies \tilde L = L + \frac{d\phi}{dt}$$

$$\implies \frac{d\phi}{dt} = \frac{\partial L}{\partial t}\epsilon$$

$$= \Big[\frac{d}{dt}L - \Big(\frac{\partial L}{\partial q}\dot q + \underbrace{\frac{\partial L}{\partial \dot q}\ddot q}_{-\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\dot q + \frac{d}{dt}\left(\dot q \frac{\partial L}{\partial \dot q}\right)}\Big)\Big]\epsilon$$

$$\implies \frac{d\phi}{dt} = \Big[\frac{d}{dt}\Big(L - \dot q \frac{\partial L}{\partial \dot q}\Big) + \underbrace{\Big(\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}\Big)}_{0 \text{ on classical trajectory}}\dot q\Big]\epsilon$$

$$\implies \frac{d}{dt}\Big(L - \dot q\frac{\partial L}{\partial \dot q} - \phi\Big) = 0$$

$$C = L - \dot q \frac{\partial L}{\partial \dot q} - \phi = L - \dot q\frac{\partial L}{\partial \dot q} - \alpha q$$

$$= F + \alpha t \dot q - \dot q\Big(\frac{\partial F}{\partial \dot q} + \alpha t\Big) - \alpha q$$

$$\boxed{C = F - \dot q \frac{\partial F}{\partial \dot q} - \alpha q}$$

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3.b (20 points) Let $F(q, \dot{q}) = \frac{m}{2}\dot{q}^2 + \alpha q$ and suppose that at $t = 0$ we have $q(0) = \beta$ and $\dot{q}(0) = \gamma$, where β and γ are real parameters. Use the constant of motion you found in part a of this problem to obtain the solution of the classical equation of motion. You should give a closed form expression for q as a function of t.

$$C = F - \dot{q}\frac{\partial F}{\partial \dot{q}} - \alpha q$$

$$= \frac{m}{2}\dot{q}^2 + \alpha q - \dot{q}(m\dot{q}) - \alpha q$$

$$= -\frac{m}{2}\dot{q}^2 = \text{const.}$$

At $t=0$, $C = -\frac{m}{2}\gamma^2$

$$\Rightarrow \dot{q}^2 = \gamma^2 \Rightarrow \begin{matrix} \dot{q} = \pm\gamma \\ \dot{q}(0) = \gamma \end{matrix} \mapsto \begin{matrix} \dot{q}(t) = \gamma \\ q(0) = \beta \end{matrix}$$

$$\Rightarrow \boxed{q(t) = \gamma t + \beta}$$