Phys 402: Midterm Exam 2

April 26, 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 (10 points) Show that the angular momentum operators $\hat{L}_1 := \hat{L}_x$, $\hat{L}_2 := \hat{L}_y$, and $\hat{L}_3 := \hat{L}_z$ satisfy the identity $[\hat{L}_i, \hat{L}_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\hat{L}_k$ for all $i,j = 1,2,3$, where ϵ_{ijk} stands for the Levi Civita symbol.

$$LHS = [\hat{L}_i, \hat{L}_j] = \Big[\sum_{k,l=1}^{3}\epsilon_{ikl}\hat{x}_k\hat{p}_l, \sum_{m,n=1}^{3}\epsilon_{jmn}\hat{x}_m\hat{p}_n\Big]$$

$$= \sum_{k,l,m,n=1}^{3}\epsilon_{ikl}\epsilon_{jmn}[\hat{x}_k\hat{p}_l, \hat{x}_m\hat{p}_n]$$

$$= \sum_{k,l,m,n=1}^{3}\epsilon_{ikl}\epsilon_{jmn}\Big(\underbrace{[\hat{x}_k, \hat{x}_m\hat{p}_n]}_{i\hbar\hat{x}_m\delta_{kn}}\hat{p}_l + \hat{x}_k\underbrace{[\hat{p}_l, \hat{x}_m\hat{p}_n]}_{-i\hbar\delta_{lm}\hat{p}_n}\Big)$$

$$= i\hbar\sum_{k,l,m=1}^{3}\underbrace{\epsilon_{ikl}\epsilon_{jmk}}_{\epsilon_{kli}\epsilon_{kjm}}\hat{x}_m\hat{p}_l - i\hbar\sum_{k,l,n=1}^{3}\underbrace{\epsilon_{ikl}\epsilon_{jln}}_{\epsilon_{lik}\epsilon_{lnj}}\hat{x}_k\hat{p}_n$$

$$= i\hbar\sum_{l,m=1}^{3}(\delta_{lj}\delta_{im} - \delta_{lm}\delta_{ij})\hat{x}_m\hat{p}_l - i\hbar\sum_{k,n=1}^{3}(\delta_{in}\delta_{kj} - \delta_{ij}\delta_{kn})\hat{x}_k\hat{p}_n$$

$$= i\hbar\Big(\hat{x}_i\hat{p}_j - \delta_{ij}\sum_{l=1}^{3}\hat{x}_l\hat{p}_l - \hat{x}_j\hat{p}_i + \delta_{ij}\sum_{k=1}^{3}\hat{x}_k\hat{p}_k\Big)$$

$$= i\hbar(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)$$

$$RHS = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\hat{L}_k = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}\sum_{l,m=1}^{3}\epsilon_{klm}\hat{x}_l\hat{p}_m = i\hbar\sum_{l,m=1}^{3}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\hat{x}_l\hat{p}_m$$

$$= i\hbar(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i) = LHS \checkmark$$

Problem 2 (15 points) Find the matrix representation of the angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in the subspace of $L^2(\mathbb{R}^3)$ where $l = 1$. Recall that

$$\hat{L}^2|l,m\rangle = l(l+1)\hbar^2\,|l,m\rangle, \qquad \hat{L}_3|l,m\rangle = m\hbar\,|l,m\rangle,$$
$$\hat{L}_\pm := \hat{L}_x \pm i\hat{L}_y, \qquad \hat{L}_\pm|l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,|l,m\pm 1\rangle.$$

$l = 1, \quad m = -1, 0, 1$

$$\hat{L}_+|1,m\rangle = \hbar\sqrt{(1-m)(2+m)}\,|1,m+1\rangle = \begin{cases} \hbar\sqrt{2}\,|1,0\rangle & \text{for } m=-1 \\ \hbar\sqrt{2}\,|1,1\rangle & \text{for } m=0 \\ 0 & \text{for } m=1 \end{cases}$$

$$\Rightarrow \mathbb{L}_+ = [\langle 1,m'|\hat{L}_+|1,m\rangle] = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{bmatrix} = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbb{L}_- = \mathbb{L}_+^\dagger = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{L}_x = \frac{1}{2}(\mathbb{L}_+ + \mathbb{L}_-) = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbb{L}_y = \frac{1}{2i}(\mathbb{L}_+ - \mathbb{L}_-) = \frac{\hbar}{\sqrt{2}\,i} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbb{L}_z = [\langle 1,m'|\hat{L}_z|1,m\rangle] = [\hbar m\,\delta_{mm'}]$$

$$= \begin{bmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar \end{bmatrix} = \hbar \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3 Consider a free particle of mass μ moving in the x-y plane so that its dynamics is described by the Hamiltonian operator $\hat{H} := \frac{1}{2\mu}(\hat{P}_x^2 + \hat{P}_y^2)$.

3.a (10 points) Derive the expression for the Laplacian in two dimensions in polar coordinates: $\rho := \sqrt{x^2 + y^2}$ and $\varphi := \tan^{-1}(y/x)$.

$$\partial_x = \frac{\partial \rho}{\partial x}\partial_\rho + \frac{\partial \varphi}{\partial x}\partial_\varphi = \frac{x}{\rho}\partial_\rho - \frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}}\partial_\varphi = \cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi$$

$$\partial_y = \frac{\partial \rho}{\partial y}\partial_\rho + \frac{\partial \varphi}{\partial y}\partial_\varphi = \frac{y}{\rho}\partial_\rho + \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}}\partial_\varphi = \sin\varphi\,\partial_\rho + \frac{\cos\varphi}{\rho}\partial_\varphi$$

$$\nabla^2 = \partial_x^2 + \partial_y^2 = \left(\cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi\right)\left(\cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi\right) +$$
$$\left(\sin\varphi\,\partial_\rho + \frac{\cos\varphi}{\rho}\partial_\varphi\right)\left(\sin\varphi\,\partial_\rho + \frac{\cos\varphi}{\rho}\partial_\varphi\right)$$

$$= \cos^2\varphi\,\partial_\rho^2 - \cos\varphi\sin\varphi\,\partial_\rho\frac{1}{\rho}\partial_\varphi - \frac{\sin\varphi}{\rho}\partial_\varphi\cos\varphi\,\partial_\rho + \frac{\sin\varphi}{\rho^2}\partial_\varphi\sin\varphi\,\partial_\varphi +$$
$$\sin^2\varphi\,\partial_\rho^2 + \sin\varphi\cos\varphi\,\partial_\rho\frac{1}{\rho}\partial_\varphi + \frac{\cos\varphi}{\rho}\partial_\varphi\sin\varphi\,\partial_\rho + \frac{\cos\varphi}{\rho^2}\partial_\varphi\cos\varphi\,\partial_\varphi$$

$$= \partial_\rho^2 + \frac{\sin^2\varphi}{\rho}\partial_\rho - \frac{\sin\varphi\cos\varphi}{\rho}\partial_\varphi\partial_\rho + \frac{\sin\varphi\cos\varphi}{\rho^2}\partial_\varphi + \frac{\sin^2\varphi}{\rho^2}\partial_\varphi^2 +$$
$$+ \frac{\cos^2\varphi}{\rho}\partial_\rho + \frac{\cos\varphi\sin\varphi}{\rho}\partial_\varphi\partial_\rho - \frac{\cos\varphi\sin\varphi}{\rho^2}\partial_\varphi + \frac{\cos^2\varphi}{\rho^2}\partial_\varphi^2$$

$$= \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2$$

3.b (5 points) Use your response to Problem 3a to show that $\hat{H} = \hat{D} + \frac{\hat{L}_z^2}{2\mu\rho^2}$ where $\hat{L}_z$ is the z-component of the angular momentum operator and $\hat{D}$ is a linear operator that in the position representation takes the form of a differential operator involving the radial coordinate ρ. Find the explicit form of this differential operator.

Note: You may use the fact that $\langle\rho,\varphi|\hat{L}_z = -i\hbar\partial_\varphi\langle\rho,\varphi|$.

$$\langle\rho,\varphi|\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2\langle\rho,\varphi| = -\frac{\hbar^2}{2\mu}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2\right]\langle\rho,\varphi|$$

$$= \langle\rho,\varphi|\left(\hat{D} + \frac{1}{2\mu\rho^2}\hat{L}_z^2\right)$$

where $(\hat{D}\psi)(\rho,\varphi) = -\frac{\hbar^2}{2\mu}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right)\psi(\rho,\varphi)$

3.c (20 points) Use the formula, $\hat{H} = \hat{D} + \frac{\hat{L}_z^2}{2\mu\hat{\rho}^2}$ for the Hamiltonian of this system to reduce the solution of its time-independent Schrödinger equation to that of a particle moving on the half-line $[0,\infty)$. Determine the Hamiltonian for this particle.

Hint: Express the energy eigenfunctions in the form $\psi_E(\rho,\varphi) = \rho^\alpha U(\rho)F(\varphi)$ for a real number α and functions U and F, and determine α and $F(\varphi)$ such that ψ_E is an eigenvector of $\hat{L}_z$ and $U(\rho)$ solves a time-independent Schrödinger equation defined in $[0,\infty)$.

$$\langle \rho,\varphi|\hat{H}|\psi_E\rangle = E\langle \rho,\varphi|\psi_E\rangle \Rightarrow [\hat{H},\hat{L}_z] = 0$$

$$\Rightarrow L_z|\psi_E\rangle = \hbar m|\psi_E\rangle \Rightarrow F(\varphi) = a\,e^{im\varphi} \qquad a\in\mathbb{C}\setminus\{0\},\ m\in\mathbb{Z}$$

$$\Downarrow$$

$$\langle \rho,\varphi|\psi_E\rangle = a\,\rho^\alpha U(\rho)\,e^{im\varphi}$$

$$\Rightarrow -\frac{\hbar^2}{2\mu}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2\right]\rho^\alpha U(\rho)\,e^{im\varphi} = E\,\rho^\alpha U(\rho)\,e^{im\varphi}$$

we know that $E>0$ because $\langle\psi|\hat{H}|\psi\rangle > 0 \Rightarrow$

let $k := \frac{\sqrt{2\mu E}}{\hbar} \Rightarrow$

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{m^2}{\rho^2}\right)\rho^\alpha U(\rho) = -k^2\rho^\alpha U(\rho)$$

$$\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \left(k^2 - \frac{m^2}{\rho^2}\right)\right]\rho^\alpha U(\rho) = 0$$

$$\partial_\rho(\rho^\alpha U) = \alpha\rho^{\alpha-1}U + \rho^\alpha U'$$

$$\partial_\rho^2(\rho^\alpha U) = \alpha(\alpha-1)\rho^{\alpha-2}U + 2\alpha\rho^{\alpha-1}U' + \rho^\alpha U''$$

$$\Rightarrow \rho^\alpha U'' + 2\alpha\rho^{\alpha-1}U' + \alpha(\alpha-1)\rho^{\alpha-2}U + \alpha\rho^{\alpha-2}U + \rho^{\alpha-1}U' + \left(k^2 - \frac{m^2}{\rho^2}\right)\rho^\alpha U = 0$$

$$\Rightarrow \rho^\alpha U'' + (2\alpha+1)\rho^{\alpha-1}U' + [\alpha^2 - \alpha + \alpha - m^2]\rho^{\alpha-2}U + k^2\rho^\alpha U = 0$$

Take $\alpha = -\frac{1}{2} \Rightarrow U'' + \frac{\frac{1}{4} - m^2}{\rho^2}U = -k^2 U$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2\mu}U'' + \frac{\hbar^2}{2\mu\rho^2}\left(m^2 - \frac{1}{4}\right)U = EU}$$

$$\hat{H}_{eff} = \frac{\hat{P}^2}{2\mu} + V(\hat{\rho}) \quad \text{with} \quad V(\rho) := \frac{\hbar^2}{2\mu}\left(m^2 - \frac{1}{4}\right)\frac{1}{\rho^2}.$$

3.d (10 points) Find the asymptotic expression (valid when $\rho \to \infty$) for the common eigenfunctions $\psi_E(\rho, \varphi)$ of $\hat{H}$ and $\hat{L}_z$.

For $\rho \to \infty$,

$$-\frac{\hbar^2}{2\mu} U'' \approx E U$$

$$\Downarrow$$

$$U'' + k^2 U \approx 0$$

$$\Downarrow$$

$$U(\rho) \approx b e^{ik\rho} + c e^{-ik\rho}$$

for some $b, c \in \mathbb{C}$.

$$\Downarrow$$

$$\psi_E(\rho, \varphi) \approx \rho^{-\frac{1}{2}} (b e^{ik\rho} + c e^{-ik\rho})(a e^{im\varphi})$$

$$\approx \left(\frac{A e^{ik\rho} + B e^{-ik\rho}}{\sqrt{\rho}} \right) e^{im\varphi}$$

when $A := ab$, $B := ac$

Problem 4 Consider a quantum system with Hilbert space $L^2(\mathbb{R}^3)$ whose Hamiltonian $\hat{H}$ differs from that of a Hydrogen atom by an additive term of the form $\omega\hat{L}_z$, i.e.,

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2\mu} - \frac{e^2}{\hat{r}} + \omega\hat{L}_z,$$

where ω is a positive real constant smaller than $10^{-6}\mathrm{Ry}/\hbar$.

4.a (20 points) Determine the first four smallest energy eigenvalues of $\hat{H}$ and their degree of degeneracy (multiplicity).

$$[\hat{L}_z, \frac{\hat{P}^2}{2\mu} - \frac{e^2}{\hat{r}}] = 0 \Rightarrow [\hat{H}, \frac{\hat{P}^2}{2\mu} - \frac{e^2}{\hat{r}}] = 0 \Rightarrow$$

$\exists$ a common complete set of eigenvectors of $\hat{H}$ that are eigenvectors of the Hydrogen atom Hamiltonian $\Rightarrow$

$$\hat{H}|n,\ell,m\rangle = E|n,\ell,m\rangle$$

$$\underbrace{(\frac{\hat{P}^2}{2\mu} - \frac{e^2}{\hat{r}})|n,\ell,m\rangle}_{-\frac{R_y}{n^2}|n,\ell,m\rangle} + \omega\hat{L}_z|n,\ell,m\rangle = E|n,\ell,m\rangle$$

$$-\frac{R_y}{n^2}|n,\ell,m\rangle + \hbar\omega m|n,\ell,m\rangle = E|n,\ell,m\rangle$$

$$\Rightarrow \boxed{E = -\frac{R_y}{n^2} + m\hbar\omega} \quad n = 1,2,\dots, \quad \begin{array}{l}\ell = 0,1,\dots,n-1\\ m = -\ell,-\ell+1,\dots,\ell\end{array}$$

Ground state: $n=1 \Rightarrow \ell=m=0 \Rightarrow \begin{cases} E_1 = -R_y \\ \text{nondegenerate.}\end{cases}$

Lowest excited states: $n = 2 \Rightarrow \begin{cases} \ell = 0 \Rightarrow m = 0 \\ \ell = 1 \Rightarrow m = -1,0,1\end{cases}$

$\ell = 1, m = -1 \Rightarrow E_2 = -\frac{R_y}{4} - \hbar\omega$, nondegenerate

$\left.\begin{array}{l}\ell = 0, m = 0\\ \ell = 1, m = 0\end{array}\right\} \Rightarrow E_3 = -\frac{R_y}{4}$: doubly degenerate

$\ell = 1, m = 1 \Rightarrow E_4 = -\frac{R_y}{4} + \hbar\omega$: nondegenerate

So the smallest energy eigenvalues are $E_1 < E_2 < E_3 < E_4$

$E_1, E_2, \& E_4$ are nondegenerate but E_3 is doubly degenerate

4.b (10 points) Find the degree of degeneracy of energy levels of $\hat{H}$ with energy smaller than -10^{-3}Ry.

For each $n = 1, 2, 3, \dots$, $l = 0, 1, \dots, n-1$ &

$m = -l, -l+1, \dots, l$

$\Downarrow$

Degeneracy of eigenvalue $E = -\frac{R_y}{n^2} + m\hbar\omega$ are given by the number of different possible values of l that yield the same value of m.

For $l = 0 \Rightarrow m = 0$

For $l = 1 \Rightarrow m = -1, 0, 1$

$\vdots$

For $l = n-1 \Rightarrow m = -n+1, -n+2, \dots, n-1$

Then are n possibilities for getting $m = 0$

" " $n-1$ " " " $m = \pm 1$

" " $n-2$ " " " $m = \pm 2$

$\vdots$

" " $n-k$ " " " $m = \pm k$

" " " " " $m = \pm(n-1)$

" is 1 "

$\Rightarrow$ $E = -\frac{R_y}{n^2} + m\hbar\omega$ is $(n-k)$ fold degenerate

for $m = \pm k$ when $k = 0, 1, \dots, n-1$.