In the following problems $\hat{X}$ and $\hat{P}$ denote the standard position and momentum operators acting in $L^2(\mathbb{R})$.

1 (5 points) Solve Exercise Problem 4.2.2 on Page 139 of Shankar.

2 (5 points) Solve Exercise Problem 4.2.3 on Page 139 of Shankar.

3 (10 points) Solve Exercise Problem 5.1.3 on Page 155 of Shankar.

4 (10 points) Solve Exercise Problem 5.2.1 on Page 163 of Shankar.

5 A particle of mass m moving in a straight line interacts with a potential of the form

$$v(x) = \left\{ \begin{array}{ccc} -V_0 & \text{for} & |x| < a, \\ 0 & \text{for} & |x| \geq a, \end{array} \right.$$

where V_0 and a are positive real parameters.

5a (30 points) Obtain all possible nonzero solutions of the time-independent Schrödinger equation and determine the structure of the energy spectrum of the system.

5.b (25 points) For each positive value of energy $E > 0$ the time-independent Schrödinger equation admits a pair of solutions $\psi_{E\pm}(x)$ satisfying

$$\psi_{E-}(x) = \left\{ \begin{array}{cc} T_r(E)e^{-ik_E x} & \text{for} \quad x < -a, \\ e^{-ik_E x} + R_r(E)e^{ik_E x} & \text{for} \quad x > a, \end{array} \right.$$

$$\psi_{E+}(x) = \left\{ \begin{array}{cc} e^{ik_E x} + R_l(E)e^{-ik_E x} & \text{for} \quad x < -a, \\ T_l(E)e^{ik_E x} & \text{for} \quad x > a, \end{array} \right.$$

where $k_E := \sqrt{2mE}/\hbar$, and R_r, T_r, R_l, and T_l are functions of E. Obtain explicit expressions for these functions, i.e., give their formula in terms of E, a, V_0 and m.

5.c (5 points) Show that $|R_l| = |R_r|$.

5.d (10 points) Show that $|R_l|^2 + |T_l|^2 = 1$ and $|R_r|^2 + |T_r|^2 = 1$.