Assume that the operators appearing in what follows are defined in the whole Hilbert space and there are no domain issues when they are composed, i.e., ignore all the domain-related problems.

1 (12 points) Let A, B, and C be linear operators acting in a Hilbert space, and $\{A, B\} := AB + BA$. Show that the following identities hold.

a) $[A, B + C] = [A, B] + [A, C]$.

b) $[A, BC] = [A, B]C + B[A, C]$.

c) $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$.

d) $[A, BC] = \{A, B\}C - B\{A, C\}$.

2 (6 points) Let A and B be Hermitian operators acting in a Hilbert space, and $\{A, B\} := AB + BA$. Show that

a) $\{A, B\}$ is Hermitian.

b) $i[A, B]$ is Hermitian.

3 (10 points) Let A be a linear operator acting in (a finite-dimensional) Hilbert space. Show that there is a unique pair of Hermitian operators B and C such that $A = B + iC$.

4 (20 points) Let $a \in \mathbb{R}$ and $\hat{\mathcal{P}}_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the parity operator defined by $\hat{\mathcal{P}}_a\psi(x) := \psi(2a - x)$. Show that

a) $\hat{\mathcal{P}}_a$ is Hermitian.

b) $\hat{\mathcal{P}}_a$ is unitary.

c) $\{\hat{X}, \hat{\mathcal{P}}_a\} = 2a\hat{\mathcal{P}}_a$.

d) $\{\hat{P}, \hat{\mathcal{P}}_a\} = 0$.

5 (12 points) Let a and $\mathcal{P}_a$ be as in Problem 4. Express the following operators as linear combinations of $\hat{\mathcal{P}}_a$, $\hat{X}$, $\hat{P}$, and the identity operators $\hat{I}$.

a) $\hat{\mathcal{P}}_a\hat{X}\,\hat{\mathcal{P}}_a$.

b) $\hat{\mathcal{P}}_a\hat{P}\,\hat{\mathcal{P}}_a$.

c) $\hat{\mathcal{P}}_a\hat{X}^2\,\hat{\mathcal{P}}_a$.

d) $\hat{\mathcal{P}}_a\hat{P}^2\,\hat{\mathcal{P}}_a$.

6 (20 points) Let a and $\hat{\mathcal{P}}_a$ be as in Problem 4. Solve the eigenvalue problem for $\hat{\mathcal{P}}_a$, i.e., find its eigenvalues and determine the form of the most general eigenvector for each eigenvalue.

7 (20 points) Let a and $\hat{\mathcal{P}}_a$ be as in Problem 4. Show that for every real number α, $e^{\alpha\hat{\mathcal{P}}_0} = \cosh\alpha\,\hat{I} + \sinh\alpha\,\hat{\mathcal{P}}_0$.