Solutions

Phys 401/OEPE 541: Midterm Exam 2

November 11, 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 Let $(V, \langle\cdot|\cdot\rangle)$ be a complex inner-product space, W be a subspace of V, and $W^{\perp} := \{a \in V \mid \text{for all } w \in W,\ \langle a|w\rangle = 0\ \}$.

1.a (3 points) Show that the inner product of every element of V with the zero vector o is zero, i.e., for all $v \in V$, $\langle v|o\rangle = 0$.

$\forall v \in V$, $\underbrace{0}_{\text{zero number}} . v = \underbrace{0}_{\text{zero vector}}$

$\Rightarrow \langle v|0\rangle = \langle v|0.v\rangle = 0\langle v|v\rangle = 0$. ▨

1.b (3 points) Show that $W^\perp$ is a subspace of V.

$\forall w \in W, \quad \langle 0|w\rangle = \langle w|0\rangle^* = 0 \Rightarrow 0 \in W^\perp$ ①

$\forall a, b \in W^\perp, \quad \forall w \in W, \quad \langle a|w\rangle = \langle b|w\rangle = 0$

$\forall \alpha, \beta \in \mathbb{C}, \quad \langle \alpha a + \beta b|w\rangle = \alpha^* \langle a|w\rangle + \beta^* \langle b|w\rangle$

$= \alpha^* \cdot 0 + \beta^* \cdot 0 = 0$

$\Rightarrow \alpha a + \beta . b \in W^\perp$ ②

①, ② $\Rightarrow W^\perp$ is a subspace of V.

1.c (9 points) Suppose that V is 4-dimensional, $B := \{b_1, b_2, b_3, b_4\}$ is an orthonormal basis of V, and $W :=$ Span$(\{b_1 + b_2, b_2 - b_3\})$. Find an orthonormal basis for W

$\langle b_1 + b_2 | b_1 + b_2\rangle = \langle b_1|b_1\rangle + \langle b_2|b_2\rangle = 2 \Rightarrow$

$\boxed{u_1 := \frac{1}{\sqrt{2}}(b_1 + b_2)}$ is a unit vector belongs to W.

let $u_2' := b_2 - b_3 - \langle u_1|b_2 - b_3\rangle u_1$

$\langle u_1|b_2 - b_3\rangle = \langle \frac{1}{\sqrt{2}}(b_1 + b_2)|b_2 - b_3\rangle = \frac{1}{\sqrt{2}}$

$\Rightarrow u_2' = b_2 - b_3 - \frac{1}{\sqrt{2}} u_1 = b_2 - b_3 - \frac{1}{2}(b_1 + b_2) = -\frac{1}{2} b_1 + \frac{1}{2} b_2 - b_3$

$\langle u_2'|u_2'\rangle = \langle -\frac{1}{2} b_1 + \frac{1}{2} b_2 - b_3 | -\frac{1}{2} b_1 + \frac{1}{2} b_2 - b_3\rangle$

$= \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$

$\Rightarrow \boxed{u_2 := \frac{1}{\|u_2'\|} u_2' = \sqrt{\frac{2}{3}}\left(-\frac{1}{2} b_1 + \frac{1}{2} b_2 - b_3\right)}$

$\{u_1, u_2\}$ is an orthonormal basis of W

Problem 2 Let V be a complex inner-product space with inner product $\langle\cdot|\cdot\rangle$, W be a subspace of V, $W^\perp := \{a \in V \mid \text{for all } w \in W,\ \langle a|w\rangle = 0\}$, $\hat{0}$ and $\hat{I}$ are respectively the zero and identity operators acting in V, and $\hat{\Pi}_\pm : V \to V$ be the linear operators with domain V that satisfy the following conditions:

1. Range of $\hat{\Pi}_+$ is W.
2. Range of $\hat{\Pi}_-$ is $W^\perp$.
3. $\hat{\Pi}_+^2 = \hat{\Pi}_+$ and $\hat{\Pi}_-^2 = \hat{\Pi}_-$
4. $\hat{\Pi}_+ + \hat{\Pi}_- = \hat{I}$.

2.a (5 points) Show that $\hat{\Pi}_+\hat{\Pi}_- = \hat{0}$.

$$\hat{\Pi}_+ + \hat{\Pi}_- = \hat{I} \Rightarrow \hat{\Pi}_+(\hat{\Pi}_+ + \hat{\Pi}_-) = \hat{\Pi}_+$$

$$\Rightarrow \underbrace{\hat{\Pi}_+^2}_{\hat{\Pi}_+} + \hat{\Pi}_+\hat{\Pi}_- = \hat{\Pi}_+$$

$$\hat{\Pi}_+ + \hat{\Pi}_+\hat{\Pi}_- = \hat{\Pi}_+ \Rightarrow \hat{\Pi}_+\hat{\Pi}_- = \hat{0}$$

2.b (5 points) Show that W is the null-space of $\hat{\Pi}_-$.

$\forall a \in W, \quad a = (\hat{\Pi}_- + \hat{\Pi}_+)a = \hat{\Pi}_- a + \hat{\Pi}_+ a$

$$\Rightarrow \hat{\Pi}_- a = \underbrace{a}_{\in W} - \underbrace{\hat{\Pi}_+ a}_{\in W} \in W$$

Also $\hat{\Pi}_- a \in \mathrm{Ran}(\hat{\Pi}_-) = W^\perp \Rightarrow \langle \hat{\Pi}_- a | \hat{\Pi}_- a\rangle = 0$

$\Rightarrow \hat{\Pi}_- a = 0 \Rightarrow a \in \mathrm{Nul}(\hat{\Pi}_-) \Rightarrow W \subseteq \mathrm{Nul}(\hat{\Pi}_-)$ (1)

Also, $\forall b \in \mathrm{Nul}(\hat{\Pi}_-)$, $\hat{\Pi}_- b = 0$

$\Rightarrow b = (\hat{\Pi}_- + \hat{\Pi}_+)b = \hat{\Pi}_- b + \hat{\Pi}_+ b = \hat{\Pi}_+ b \in W$

$\Rightarrow \mathrm{Nul}(\hat{\Pi}_-) \subseteq W$ (2)

(1) & (2) $\Rightarrow W = \mathrm{Nul}(\hat{\Pi}_-)$

2.c (5 points) Show that $\hat{\Pi}_+$ and $\hat{\Pi}_-$ are Hermitian operators.

$\forall a, b \in V, \quad a = (\hat{\Pi}_+ + \hat{\Pi}_-) a = \hat{\Pi}_+ a + \hat{\Pi}_- a$

$$\Rightarrow \langle a | \hat{\Pi}_+ b \rangle = \langle \hat{\Pi}_+ a + \hat{\Pi}_- a | \hat{\Pi}_+ b \rangle$$

$$= \langle \hat{\Pi}_+ a | \hat{\Pi}_+ b \rangle + \langle \underset{\uparrow \, W^\perp}{\hat{\Pi}_- a} | \underset{\uparrow \, W}{\hat{\Pi}_+ b} \rangle \nearrow 0$$

$$\underset{①}{=} \langle \hat{\Pi}_+ a | \hat{\Pi}_- b \rangle$$

Similarly

$$\langle \hat{\Pi}_+ a | b \rangle = \langle \hat{\Pi}_+ a | \hat{\Pi}_+ b + \hat{\Pi}_- b \rangle$$

$$= \langle \hat{\Pi}_+ a | \hat{\Pi}_+ b \rangle + \langle \underset{\uparrow \, W}{\hat{\Pi}_+ a} | \underset{\uparrow \, W^\perp}{\hat{\Pi}_- b} \rangle \nearrow 0$$

$$\underset{②}{=} \langle \hat{\Pi}_+ a | \hat{\Pi}_+ b \rangle$$

$① \, \& \, ② \Rightarrow \Pi_+^\dagger = \Pi_+$

$$\Rightarrow \hat{\Pi}_-^\dagger = (\hat{I} - \hat{\Pi}_+)^\dagger = \hat{I}^\dagger - \hat{\Pi}_+^\dagger$$

$$= \hat{I} - \hat{\Pi}_+$$

$$= \hat{\Pi}_- \quad \blacksquare$$

Problem 3 Let V, B, and W be as in Problem 1.c., i.e., $B := \{b_1, b_2, b_3, b_4\}$ is an orthonormal basis of V, and $W := \mathrm{Span}(\{b_1 + b_2, b_2 - b_3\})$, and $\hat{\Pi}_\pm$ are the linear operators defined in Problem 2.

3.a (8 points) Express $\hat{\Pi}_+$ in the form $\sum_{i,j=1}^{4} P_{ij}|b_i\rangle\langle b_j|$ where P_{ij} are complex numbers. You are asked to find these numbers.

$\hat{\Pi}_+$ is the orthogonal pojection op. onto W $\Rightarrow$

$\hat{\Pi}_+ = |w_1\rangle\langle w_1| + |w_2\rangle\langle w_2|$ where $\{w_1, w_2\}$ is the orthonormal basis of W constructed in Problem 1.c.

$$\Rightarrow \hat{\Pi}_+ = \frac{1}{2}(|b_1\rangle + |b_2\rangle)(\langle b_1| + \langle b_2|) +$$

$$\frac{2}{3}\left(-\frac{1}{2}|b_1\rangle + \frac{1}{2}|b_2\rangle - |b_3\rangle\right)\left(-\frac{1}{2}\langle b_1| + \frac{1}{2}\langle b_2| - \langle b_3|\right)$$

$$= \left(\frac{1}{2} + \frac{1}{6}\right)|b_1\rangle\langle b_1| + \left(\frac{1}{2} - \frac{1}{6}\right)|b_1\rangle\langle b_2| + \frac{1}{3}|b_1\rangle\langle b_3| +$$

$$\left(\frac{1}{2} - \frac{1}{6}\right)|b_2\rangle\langle b_1| + \left(\frac{1}{2} + \frac{1}{6}\right)|b_2\rangle\langle b_2| + \left(-\frac{1}{3}\right)|b_2\rangle\langle b_3| +$$

$$\left(\frac{1}{3}\right)|b_3\rangle\langle b_1| + \left(-\frac{1}{3}\right)|b_3\rangle\langle b_2| + \frac{2}{3}|b_3\rangle\langle b_3|$$

$$= \frac{2}{3}|b_1\rangle\langle b_1| + \frac{1}{3}(|b_1\rangle\langle b_2| + |b_2\rangle\langle b_1|) + \frac{1}{3}(|b_1\rangle\langle b_3| + |b_3\rangle\langle b_1|)$$

$$+ \frac{2}{3}|b_2\rangle\langle b_2| - \frac{1}{3}(|b_2\rangle\langle b_3| + |b_3\rangle\langle b_2|) + \frac{2}{3}|b_3\rangle\langle b_3|$$

3.b (4 points) Compute $(2\hat{\Pi}_+ - 3\hat{\Pi}_-)b_2$.

$$(2\hat{\Pi}_+ - 3\hat{\Pi}_-)b_2 = (2\hat{\Pi}_+ - 3(\hat{I} - \hat{\Pi}_+))b_2 = (5\hat{\Pi}_+ - 3\hat{I})b_2$$

$$\hat{\Pi}_+|b_2\rangle = \frac{1}{3}|b_1\rangle + \frac{2}{3}|b_2\rangle - \frac{1}{3}|b_3\rangle$$

$$\Rightarrow (2\hat{\Pi}_+ - 3\hat{\Pi}_-)b_2 = \frac{5}{3}b_1 + \frac{10}{3}b_2 - \frac{5}{3}b_3 - 3b_2$$

$$= \frac{5}{3}b_1 + \frac{1}{3}b_2 - \frac{5}{3}b_3$$

$$= \frac{1}{3}(5b_1 + b_2 - 5b_3)$$

3.c (8 points) Find an orthonormal basis for $W^{\perp}$.

$W^{\perp} = \mathrm{Ran}(\hat{\Pi}_-)$

$\hat{\Pi}_- b_2 = (\hat{I} - \hat{\Pi}_+) b_2 = b_2 - (\frac{1}{3} b_1 + \frac{2}{3} b_2 - \frac{1}{3} b_3)$

$= -\frac{1}{3} b_1 + \frac{1}{3} b_2 + \frac{1}{3} b_3$

$= \frac{-1}{3} (b_1 - b_2 - b_3)$

$\Rightarrow \boxed{b_1 - b_2 - b_3 \in W^{\perp}}$ 1

Furthermore $\forall w \in W = \mathrm{Span}\{b_1 + b_2, b_2 - b_3\}$,

$\langle b_4 | w \rangle = 0$ because $\langle b_4 | b_1 + b_2 \rangle = \langle b_4 | b_2 - b_3 \rangle = 0$

$\Rightarrow \boxed{b_4 \in W^{\perp}}$

So let $a_1 := \frac{b_1 - b_2 - b_3}{\| b_1 - b_2 - b_3 \|} = \frac{1}{\sqrt{3}}(b_1 - b_2 - b_3)$ and

$a_2 := b_4$

Then $\{a_1, a_2\}$ is an orthonormal basis of $W^{\perp}$.

Problem 4 A quantum system is described by a three-dimensional Hilbert space $\mathscr{H}$ with inner product $\langle\cdot|\cdot\rangle$. Suppose that $B := \{b_1, b_2, b_3\}$ is an orthonormal basis of $\mathscr{H}$, $\psi := 2b_1 + ib_2 + b_3$, λ_ψ is the state defined by the state vector ψ, and $\hat{A}, \hat{B}, \hat{C} : \mathscr{H} \to \mathscr{H}$ be linear operators defined in $\mathscr{H}$ according to $\hat{A} := |b_1\rangle\langle b_3| - 2i|b_3\rangle\langle b_1|$ and $\hat{B} = \hat{A}^\dagger\hat{A}$ and $\hat{C} = \hat{A} + \hat{A}^\dagger + |\psi\rangle\langle\psi|$.

4.a (15 points) Compute possible values one can obtain, if one measures $\hat{B}$ in the state λ_ψ and determine the probability of measuring these values.

$$\hat{B} = (|b_3\rangle\langle b_1| + 2i|b_1\rangle\langle b_3|)(|b_1\rangle\langle b_3| - 2i|b_3\rangle\langle b_1|)$$

$$= |b_3\rangle\langle b_3| + 4|b_1\rangle\langle b_1|$$

Clear

$$\hat{B}|b_1\rangle = 4|b_1\rangle$$
$$\hat{B}|b_3\rangle = |b_3\rangle$$
$$\hat{B}|b_2\rangle = 0$$

$\Rightarrow$ Eigenvalues of $\hat{B}$ are, 4, 0, and 1 with eigenvectors $|b_1\rangle$, $|b_2\rangle$, and $|b_3\rangle$ respectively.

Possible values of measuring $\hat{B}$ are:

1) $\beta_1 = 4$ with probability:

$$\mathcal{P}_{\beta_1}(\lambda_\psi) = \frac{\langle\psi|b_1\rangle\langle b_1|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{|\langle b_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{4}{4+1+1} = \frac{4}{6} = \frac{2}{3}$$

2) $\beta_2 = 0$ with probability:

$$\mathcal{P}_{\beta_2}(\lambda_\psi) = \frac{\langle\psi|b_2\rangle\langle b_2|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{|\langle b_2|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{|i|^2}{6} = \frac{1}{6}$$

3) $\beta_3 = 1$ with probability

$$\mathcal{P}_{\beta_3}(\lambda_\psi) = \frac{\langle\psi|b_3\rangle\langle b_3|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{|\langle b_3|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{1}{6}$$

4.b (5 points) Compute the expectation value of $\hat{B}$ in the state λ_ψ.

$$\langle \hat{B} \rangle_{\lambda_+} = \frac{2}{3} \cdot 4 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 1 = \frac{16+1}{6} = \frac{17}{6}$$

we can also compute it as follows

$$\langle \hat{B} \rangle_{\lambda_+} = \frac{\langle + | \hat{B} | + \rangle}{\langle + | + \rangle} \qquad \hat{B} | + \rangle = \left(|b_3\rangle\langle b_3| + 4|b_1\rangle\langle b_1| \right)\left(2|b_1\rangle + i|b_2\rangle + |b_3\rangle \right).$$

$$\Rightarrow \hat{B}|+\rangle = |b_3\rangle + 8|b_1\rangle$$

$$\Rightarrow \langle + | \hat{B} | + \rangle = \left(2\langle b_1| - i\langle b_2| + \langle b_3| \right)\left(|b_3\rangle + 8|b_1\rangle \right) = 16 + 1 = 17$$

$$\langle + | + \rangle = 6$$

$$\Rightarrow \langle \hat{B} \rangle_{\lambda_+} = \frac{17}{6} \quad \checkmark$$

4.c (5 points) Compute the uncertainty in measuring $\hat{B}$ in the state λ_ψ.

$$\Delta \hat{B}_{\lambda_+} = \sqrt{\langle \hat{B}^2 \rangle_{\lambda_+} - \langle \hat{B} \rangle_{\lambda_+}^2}$$

$$\langle \hat{B}^2 \rangle_{\lambda_+} = \frac{\langle + | \hat{B}^2 | + \rangle}{\langle + | + \rangle} = \frac{\langle \hat{B}+ | \hat{B}+ \rangle}{\langle + | + \rangle} = \frac{\| \hat{B}+ \|^2}{\| + \|^2} = \frac{1+64}{6} = \frac{65}{6}$$

$$\langle \hat{B} \rangle_{\lambda_+}^2 = \left(\frac{17}{6} \right)^2 = \frac{289}{36}$$

$$\Rightarrow \Delta \hat{B}_{\lambda_+} = \sqrt{\frac{65}{6} - \frac{289}{36}} = \frac{1}{6}\sqrt{390 - 289} = \frac{\sqrt{101}}{6} \approx \frac{5}{3} \approx 1.67$$

4.d (15 points) Suppose that we measure $\hat{B}$ when the system is in the state λ_ψ and then measure $\hat{C}$. Supposing that the time between the two measurements is negligibly small, find the expectation value of the outcome of the second measurement.

$\hat{C} = \hat{A} + \hat{A}^\dagger + |+\rangle\langle +|$ $\quad\quad$ $\hat{A} = |b_1\rangle\langle b_3| - 2i|b_3\rangle\langle b_1|$

Expectation value of second measure ($\mathcal{E}$) is sum of possible values of the measurement γ_i times their probabilities $\mathfrak{P}_i$, i.e., $\sum_i \mathfrak{P}_i \gamma_i$ if we know the state in which $\hat{C}$ is measured. If Π_i^C is the orthogonal projection onto the eignspace of γ_i, $\mathfrak{P}_i = \dfrac{\langle\phi|\Pi_i^C\phi\rangle}{\langle\phi|\phi\rangle}$ when ϕ is a state vector in which $\hat{C}$ is measured. $\phi \in \Pi_j^B\psi$ where Π_j^B are the orthogonal projection operator on the eignspace $V_j^B = \text{null}(\hat{B} - \beta_j \hat{I})$

$\phi = \phi_1 := \Pi_1^B\psi$ with probability $\frac{2}{3}$

$\phi = \phi_2 := \Pi_2^B\psi$ " " $\frac{1}{6}$

$\phi = \phi_3 := \Pi_3^B\psi$ " " $\frac{1}{6}$.

So the exp. value of the second measurement is

$$\mathcal{E} = \sum_i \left[\frac{2}{3} \frac{\langle\phi_1|\Pi_i^C\phi_1\rangle}{\langle\phi_1|\phi_1\rangle} + \frac{1}{6} \frac{\langle\phi_2|\Pi_i^C|\phi_2\rangle}{\langle\phi_2|\phi_2\rangle} + \frac{1}{6} \frac{\langle\phi_3|\Pi_i^C|\phi_3\rangle}{\langle\phi_3|\phi_3\rangle} \right] \gamma_i$$

$$= \frac{2}{3}\langle\hat{C}\rangle_{\lambda_{\phi_1}} + \frac{1}{6}\langle\hat{C}\rangle_{\lambda_{\phi_2}} + \frac{1}{6}\langle\hat{C}\rangle_{\lambda_{\phi_3}}$$

$|\phi_1\rangle = |b_1\rangle\langle b_1|\psi\rangle = 2|b_1\rangle$

$|\phi_2\rangle = |b_2\rangle\langle b_2|\psi\rangle = -i|b_2\rangle$

$|\phi_3\rangle = |b_3\rangle\langle b_3|\psi\rangle = |b_3\rangle$

$\Downarrow$

$$\Downarrow$$

$$\langle \phi_1 | \hat{C} | \phi_1 \rangle = 4 \langle b_1 | \left(\hat{A} + \hat{A}^{\dagger} + |+\rangle\langle +| \right) | b_1 \rangle$$

$$= 4 \left[\underbrace{\langle b_1 | \hat{A} | b_1 \rangle}_{0} + \underbrace{\langle b_1 | \hat{A}^{\dagger} | b_1 \rangle}_{0} + \underbrace{|\langle b_1 | + \rangle|^2}_{2} \right]$$

$$= 16$$

$$\langle \phi_2 | \hat{C} | \phi_2 \rangle = \langle b_2 | \left[\underbrace{\langle b_2 | \hat{A} | b_2 \rangle}_{0} + \underbrace{\langle b_2 | \hat{A}^{\dagger} | b_2 \rangle}_{0} + \underbrace{|\langle b_2 | + \rangle|^2}_{\vdots} \right]$$

$$= 1$$

$$\langle \phi_3 | \hat{C} | \phi_3 \rangle = \langle b_3 | \left[\underbrace{\langle b_3 | \hat{A} | b_3 \rangle}_{0} + \underbrace{\langle b_3 | \hat{A}^{\dagger} | b_3 \rangle}_{0} + \underbrace{|\langle b_3 | + \rangle|^2}_{1} \right]$$

$$= 1$$

$$\langle \phi_1 | \phi_1 \rangle = 4, \qquad \langle \phi_2 | \phi_2 \rangle = \langle \phi_3 | \phi_3 \rangle = 1$$

$$\langle \hat{C} \rangle_{\lambda_{\phi_1}} = \frac{16}{4} = 4, \qquad \langle \hat{C} \rangle_{\lambda_{\phi_2}} = \langle \hat{C} \rangle_{\lambda_{\phi_3}} = 1$$

$$\Rightarrow \boxed{\mathcal{E} = \frac{2}{3} \cdot 4 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 = 3}$$

Problem 5 (10 points) Consider a quantum system with a finite-dimensional Hilbert space $\mathscr{H}$, and suppose that we measure an observable given by a linear operator $\hat{O}$ acting in $\mathscr{H}$. Show that the only way in which we can know the outcome of measuring $\hat{O}$ in a state λ before the measurement is that λ is an eigenstate of $\hat{O}$, i.e., $\lambda = \mathrm{Span}(\{\psi\})$, where ψ is an eigenvector of $\hat{O}$. Let $\psi \in \lambda \setminus \{0\}$ so that $\lambda = \lambda_\psi$.

$$\Delta \hat{O}_{\lambda_\psi} = 0 \iff \frac{\langle\psi|(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})^2|\psi\rangle}{\langle\psi|\psi\rangle} = 0$$

$$\iff 0 = \langle\psi|(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})|\psi\rangle$$

$$= \langle(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})\psi|(\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})\psi\rangle$$

$\hookrightarrow$ because $\langle\hat{O}\rangle_{\lambda_\psi} \in \mathbb{R} \Rightarrow \hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I}$ is Hermit.

$$\iff (\hat{O} - \langle\hat{O}\rangle_{\lambda_\psi}\hat{I})|\psi\rangle = 0$$

$$\iff \hat{O}|\psi\rangle = \langle\hat{O}\rangle_{\lambda_\psi}|\psi\rangle$$

$|\psi\rangle \neq 0$ $\hookrightarrow$ $|\psi\rangle$ is an eigenvector.

$\langle\hat{O}\rangle_{\lambda_\psi} \in \mathbb{R}$

$$\Downarrow$$

$\lambda = \lambda_\psi$ is an eigenstate.