Sol. to Prob. 3:

3.a) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB_0}{2c}(-\dot{x}y + \dot{y}x)$

$\frac{\partial L}{\partial x} = \frac{qB_0}{2c}\dot{y}$, $\quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{qB_0}{2c}y$

Let $\boxed{\alpha := \frac{qB_0}{2c}}$ $\Rightarrow$ $\frac{\partial L}{\partial x} = \alpha\dot{y}$, $\quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \alpha y$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} - \alpha\dot{y} - \alpha\dot{y} = 0$

$\Rightarrow m\ddot{x} - 2\alpha\dot{y} = 0$

$\Rightarrow \boxed{\ddot{x} - \frac{2\alpha}{m}\dot{y} = 0}$ (3.1)

$\frac{\partial L}{\partial y} = -\frac{qB_0}{2c}\dot{x} = -\alpha\dot{x}$, $\quad \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{qB_0}{2c}x = m\dot{y} + \alpha x$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \Rightarrow m\ddot{y} + \alpha\dot{x} + \alpha\dot{x} = 0$

$\Rightarrow \boxed{\ddot{y} + \frac{2\alpha}{m}\dot{x} = 0}$ (3.2)

$\frac{\partial L}{\partial z} = 0$, $\quad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \Rightarrow m\ddot{z} = 0 \Rightarrow \boxed{\ddot{z} = 0}$ (3.3)

3.b) (3.1 - 3.3) $\Rightarrow$
$$\begin{cases} \dot{x} - \frac{2\alpha}{m}y = C_1 \\ \dot{y} + \frac{2\alpha}{m}x = C_2 \\ z = v_0 t + z_0 \end{cases} \quad (3.4)$$

when C_1, C_2, v_0, z_0 are constants.

Setting $t=0$ we find from (3.4):

$$\begin{cases} 0 - 0 = C_1 & \Rightarrow C_1 = 0 \\ b + \frac{2\alpha}{m}l = C_2 & \Rightarrow C_2 = -\frac{qB_0 l}{mc} + \frac{2l}{m}\left(\frac{qB_0}{2c}\right) = 0 \\ l = z_0, \quad v_0 = a & \Rightarrow \boxed{z = at + l} \end{cases} \quad (3.5)$$

$\Downarrow$

$$\Rightarrow \begin{cases} \dot{x} - \frac{2\alpha}{m} y = 0 \\ \dot{y} + \frac{2\alpha}{m} x = 0 \\ z = at + l \end{cases} \Rightarrow \boxed{y = \frac{m}{2\alpha}\dot{x}} \quad (3.5)$$

$$\hookrightarrow \frac{m}{2\alpha}\ddot{x} + \frac{2\alpha}{m} x = 0$$

$$\Downarrow$$

$$\boxed{\ddot{x} + \left(\frac{2\alpha}{m}\right)^2 x = 0} \quad (3.6)$$

let $\omega := \left|\frac{2\alpha}{m}\right| = \left|\frac{qB_0}{mc}\right|$

$\Rightarrow x(t) = A\cos(\omega t) + B\sin(\omega t)$

$\dot{x}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$

$x(0) = l, \quad \dot{x}(0) = 0 \Longrightarrow \begin{cases} \boxed{l = A} \\ 0 = B\omega \Rightarrow \boxed{B = 0} \end{cases}$

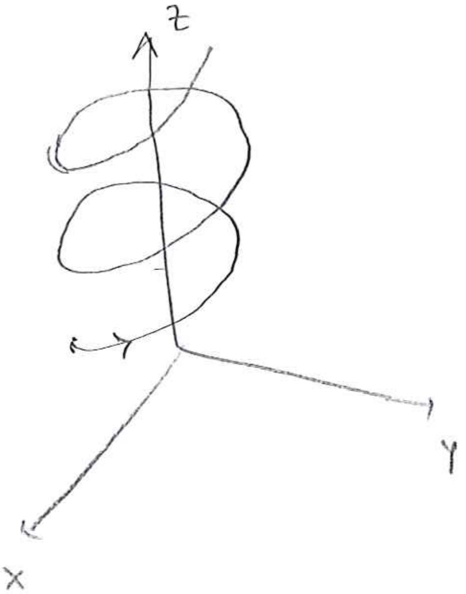
So $\boxed{x(t) = l\cos\omega t}$

$\dot{x}(t) = -l\omega\sin\omega t \Rightarrow y(t) = -\frac{ml\omega}{2\alpha}\sin\omega t$

$\Rightarrow y(t) = -\frac{ml}{2}\left(\frac{2}{m}\right)\sin\omega t \Rightarrow \boxed{y(t) = -l\sin\omega t}$

$\boxed{z = at + l}$

The trajectory is a helix.



Sol. to Prob 4:

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \alpha y \Rightarrow \dot{x} = \frac{P_x + \alpha y}{m}$$

$$P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \alpha x \Rightarrow \dot{y} = \frac{P_y - \alpha x}{m}$$

$$P_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{P_z}{m}$$

$$\boxed{H = \dot{x}P_x + \dot{y}P_y + \dot{z}P_z - L}$$

$$= \frac{P_x(P_x+\alpha y) + P_y(P_y - \alpha y) + P_z^2}{m}$$

$$- \frac{m}{2}\left(\frac{1}{m^2}\right)\left[(P_x+\alpha y)^2 + (P_y - \alpha x)^2 + P_z^2\right]$$

$$- \frac{\alpha}{m}\left[-(P_x+\alpha y)y + (P_y - \alpha x)x\right]$$

$$= \frac{1}{2m}\Big[(P_x+\alpha y)(2P_x - P_x - \alpha y + 2\alpha y) + (P_y - \alpha x)(2P_y - P_y + \alpha x - 2\alpha x) + P_z(2P_z - P_z)\Big]$$

$$\boxed{= \frac{1}{2m}\left[(P_x+\alpha y)^2 + (P_y - \alpha x)^2 + P_z^2\right]}$$

$$L_z = xP_y - yP_x \qquad \frac{d}{dt}L_z = \{L_z, H\}_{PB} = \{xP_y - yP_x, H\}_{PB}$$

$$\Rightarrow \frac{dL_z}{dt} = \{x,H\}_{PB}P_y + x\{P_y, H\}_{PB} - \{y,H\}_{PB}P_x - y\{P_x,H\}_{PB}$$

(4.1)

$$\{x,H\}_{PB} = \dot{x} = \frac{1}{m}(P_x+\alpha y), \quad \{y,H\}_{PB} = \dot{y} = \frac{1}{m}(P_y - \alpha x)$$

$$\{P_x, H\}_{PB} = \frac{1}{2m}\{P_x, (P_y - \alpha x)^2\} = -\frac{1}{2m}\frac{\partial}{\partial x}(P_y - \alpha x)^2 = \frac{\alpha}{m}(P_y - \alpha x)$$

$$\{P_y, H\}_{PB} = \frac{1}{2m}\{P_y, (P_x+\alpha y)^2\} = -\frac{1}{2m}\frac{\partial}{\partial y}(P_x+\alpha y)^2 = -\frac{\alpha}{m}(P_x+\alpha y)$$

4.1 $\hookrightarrow$ $$\boxed{\frac{dL_z}{dt}} = \frac{1}{m}(P_x+\alpha y)P_y - \frac{\alpha x}{m}(P_x+\alpha y) - \frac{1}{m}(P_y - \alpha x)P_x - \frac{\alpha y}{m}(P_y - \alpha x)$$

$$= \frac{1}{m}(P_x+\alpha y)(P_y - \alpha x) - \frac{1}{m}(P_y - \alpha x)(P_x + \alpha y)$$

$$\boxed{= 0}$$

Sol. to Prob. 5 :

5.a)

$$1 = \{\tilde{q}, \tilde{p}\}_{PB} = \frac{\partial}{\partial q}(\alpha q + \beta p)\overset{0}{\frac{\partial}{\partial p}(\gamma q + \delta)} - \underbrace{\frac{\partial}{\partial p}(\alpha q + \beta p)}_{\beta}\underbrace{\frac{\partial}{\partial q}(\gamma q + \delta)}_{\gamma}$$

$$= -\beta\gamma$$

$$\Rightarrow \quad \boxed{\beta\gamma = -1} \quad \Rightarrow \quad \gamma = -\frac{1}{\beta} \quad \Rightarrow \quad \dot{\gamma} = \frac{\dot{\beta}}{\beta^2}$$

5.b

$$\tilde{q} = \alpha q + \beta p \quad \Rightarrow \quad p = \frac{1}{\beta}(\tilde{q} - \alpha q) \quad \Rightarrow \quad p = \frac{1}{\beta}\left(\tilde{q} - \frac{\alpha}{\gamma}(\tilde{p} - \delta)\right)$$

$$\tilde{p} = \gamma q + \delta \quad \Rightarrow \quad q = \frac{1}{\gamma}(\tilde{p} - \delta)$$

$$\Downarrow$$

$$\boxed{q = -\beta\tilde{p} + \beta\delta}$$

$$p = \frac{1}{\beta}\tilde{q} - \frac{\alpha}{\beta\gamma}\tilde{p} + \frac{\delta}{\beta}$$

$$\Downarrow$$

$$\boxed{p = \frac{1}{\beta}\tilde{q} + \alpha\tilde{p} + \frac{\delta}{\beta}}$$

$$\dot{\tilde{q}} = \alpha\dot{q} + \beta\dot{p} + \dot{\alpha}q + \dot{\beta}p$$

$$\dot{\tilde{p}} = \gamma\dot{q} + \dot{\gamma}q + \dot{\delta} = -\frac{1}{\beta}\dot{q} + \frac{\dot{\beta}}{\beta^2}q + \dot{\delta}$$

$$H = \frac{p^2}{2m} \quad \Rightarrow \quad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = 0$$

$$\Rightarrow \quad \dot{\tilde{q}} = \frac{\alpha}{m}p + \dot{\alpha}q + \dot{\beta}p = \left(\frac{\alpha}{m} + \dot{\beta}\right)p + \dot{\alpha}q$$

$$= \left(\frac{\alpha}{m} + \dot{\beta}\right)\left(\frac{1}{\beta}\tilde{q} + \alpha\tilde{p} + \frac{\delta}{\beta}\right) + \dot{\alpha}(-\beta\tilde{p} + \beta\delta)$$

$$= \underbrace{\left(\frac{\alpha}{m\beta} + \frac{\dot{\beta}}{\beta}\right)}_{f_1}\tilde{q} + \underbrace{\left(\frac{\alpha^2}{m} + \alpha\dot{\beta} - \beta\dot{\alpha}\right)}_{f_2}\tilde{p} + \underbrace{\left(\frac{\alpha}{m} + \dot{\beta}\right)\left(\frac{\delta}{\beta}\right) + \beta\delta\dot{\alpha}}_{f_3}$$

$$\dot{\tilde{p}} = -\frac{1}{\beta}\left(\frac{p}{m}\right) + \frac{\dot{\beta}}{\beta^2}q + \dot{\delta} = -\frac{1}{m\beta}\left(\frac{1}{\beta}\tilde{q} + \alpha\tilde{p} + \frac{\delta}{\beta}\right) + \left(\frac{\dot{\beta}}{\beta^2}\right)(-\beta\tilde{p} + \beta\delta) + \dot{\delta}$$

$$= \underbrace{\left(-\frac{1}{m\beta^2}\right)}_{g_1}\tilde{q} + \underbrace{\left(-\frac{\alpha}{m\beta} - \frac{\dot{\beta}}{\beta}\right)}_{g_2 = -f_1}\tilde{p} + \underbrace{\left(-\frac{\delta}{m\beta^2} + \frac{\dot{\beta}\delta}{\beta} + \dot{\delta}\right)}_{g_3}$$

$$\Downarrow$$

$$\Downarrow$$

$$\dot{\tilde{q}} = f_1 \tilde{q} + f_2 \tilde{p} + f_3 = \frac{\partial \tilde{H}}{\partial \tilde{p}} \qquad (5.1)$$

$$\dot{\tilde{p}} = \underbrace{g_1}_{}\tilde{q} + \underset{\substack{\parallel \\ -f_1}}{g_2}\tilde{p} + g_3 = - \frac{\partial \tilde{H}}{\partial \tilde{q}} \qquad (5.2)$$

$$(5.1) \Rightarrow \quad \tilde{H} = \frac{1}{2} f_2 \tilde{p}^2 + (f_1 \tilde{q} + f_3)\tilde{p} + h(\tilde{q}, t)$$

$$\Rightarrow \quad \frac{\partial \tilde{H}}{\partial \tilde{q}} = f_1 \tilde{p} + \frac{\partial}{\partial \tilde{q}} h \qquad (5.3)$$

$$(5.2, 5.3) \Rightarrow \quad g_1 \tilde{q} - f_1 \tilde{p} + g_3 = - f_1 \tilde{p} - \frac{\partial}{\partial \tilde{q}} h$$

$$\Rightarrow \quad \frac{\partial}{\partial \tilde{q}} h(\tilde{q}, t) = - g_1 \tilde{q} - g_3$$

$$\Rightarrow \quad h(\tilde{q}, t) = - \frac{g_1}{2} \tilde{q}^2 + g_3 \tilde{q} + h_0(t)$$

$$\Downarrow$$

$$\boxed{\tilde{H} = \frac{1}{2} f_2 \tilde{p}^2 + (f_1 \tilde{q} + f_3)\tilde{p} - \frac{g_1}{2} \tilde{q}^2 + g_3 \tilde{q} + h_0}$$

when h_0 is a function of t.

$$f_1 := \frac{\alpha}{m\beta} + \frac{\dot{\beta}}{\beta}, \qquad f_2 := \frac{\alpha^2}{m} + \alpha\dot{\beta} - \beta\dot{\alpha}$$

$$f_3 := \frac{\alpha\delta}{m\beta} + \frac{\delta\dot{\beta}}{\beta} + \beta\delta\dot{\alpha}, \qquad g_1 := -\frac{1}{m\beta^2}, \quad \&$$

$$g_3 := -\frac{\delta}{m\beta^2} + \frac{\dot{\beta}\delta}{\beta} + \dot{\delta}$$

Sol. to Prob. 6

$$H = f(\underbrace{x_1 + 3x_2}_{u}, \underbrace{2p_1 - p_2}_{v}) \qquad f = f(u, v)$$

$$I := (x_1 + 2x_2)(3p_1 - p_2)$$

$$\frac{dI}{dt} = \{I, H\}_{PB} = \{(x_1 + 2x_2)(3p_1 - p_2), f(x_1 + 3x_2, 2p_1 - p_2)\}_{PB}$$

$$= \underbrace{\{x_1 + 2x_2, f\}_{PB}}_{\{x_1, f\} + 2\{x_2, f\}}(3p_1 - p_2) + (x_1 + 2x_2)\underbrace{\{3p_1 - p_2, f\}_{PB}}_{3\{p_1, f\} - \{p_2, f\}}$$

$$\{x_1, f\} + 2\{x_2, f\} = \frac{\partial f}{\partial p_1} + 2\frac{\partial f}{\partial p_2} = 2\frac{\partial f}{\partial v} + 2\left(-\frac{\partial f}{\partial v}\right) = 0$$

$$3\{p_1, f\} - \{p_2, f\} = -3\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = \underbrace{-3\left(\frac{\partial f}{\partial u}\right) + 3\frac{\partial f}{\partial u}}_{=0}$$

$$\equiv 0.$$