Phys 401/OEPE 541: Midterm Exam 2

November 11, 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

Estimated Grade:	
Actual Grade:	
Adjusted Grade:	

Problem 1 (10 points) Let V be a complex inner-product space with inner product $\langle\cdot|\cdot\rangle$, and $L : V \to V$ be a linear operator with domain V. The adjoint of L is the operator $L^\dagger : V \to V$ such that for all $v_1, v_2 \in V$, $\langle v_1|Lv_2\rangle = \langle L^\dagger v_1|v_2\rangle$. Prove that $L^\dagger$ is unique, i.e., if there is a linear operator $L : V \to V$ such that $\langle v_1|Lv_2\rangle = \langle Jv_1|v_2\rangle$ for all $v_1, v_2 \in V$, then $J = L^\dagger$.

Note: To get full credit, you must explain the reasoning for each step of your proof.

$\langle v_1 | L v_2\rangle = \langle L^\dagger v_1 | v_2\rangle \Rightarrow \langle L^\dagger v_1 | v_2\rangle = \langle J v_1 | v_2\rangle$

$\langle v_1 | L v_2\rangle = \langle J v_1 | v_2\rangle$

$\Rightarrow \langle L^\dagger v_1 - J v_1 | v_2\rangle = 0 \Rightarrow \langle (L^\dagger - J) v_1 | v_2\rangle = 0$

Choose $v_2 = (L^\dagger - J) v_1 \Rightarrow \langle v_2 | v_2\rangle = 0 \Rightarrow v_2 = 0$

$\Rightarrow (L^\dagger - J) v_1 = 0 \Rightarrow L^\dagger v_1 = J v_1$. But $v_1 \in V$ is arbitrary $\Rightarrow L^\dagger = J$. ∎

Problem 2 Let for all $w_1, w_2, z_1, z_2 \in \mathbb{C}$,

$$\prec (w_1, w_2), (z_1, z_2) \succ := 2(w_1^* z_1 + w_2^* z_2) - i(w_1^* z_2 - w_2^* z_1).$$

One can show that $\prec \cdot, \cdot \succ : \mathbb{C}^2 \to \mathbb{C}^2$ is an inner product on $\mathbb{C}^2$. Let V be the inner-product space obtained by endowing $\mathbb{C}^2$ with this inner product.

2.a (5 points) Find a positive real number α such that $b_1 := \alpha(1,0)$ is a unit vector in V.

$$1 = \langle b_1, b_1 \rangle = |\alpha|^2 \langle (1,0) | (1,0) \rangle = |\alpha|^2 [2] \Rightarrow |\alpha| = \frac{1}{\sqrt{2}}$$

$\Rightarrow \alpha = \frac{e^{i\varphi}}{\sqrt{2}}$ when $\varphi \in \mathbb{R}$. for example can choose $\varphi = 0$

$$\Rightarrow \boxed{\alpha = \frac{1}{\sqrt{2}}} \Rightarrow b_1 = \frac{1}{\sqrt{2}}(1,0)$$

2.b (15 points) Find a $b_2 \in \mathbb{C}^2$ such that (b_1, b_2) is an orthonormal basis of V.

let $e_2 := (0,1)$ & $\tilde{b}_2 := e_2 - \langle b_1 | e_2 \rangle b_1$

$$\langle b_1 | e_2 \rangle = \langle \tfrac{1}{\sqrt{2}}(1,0) | (0,1) \rangle = \frac{1}{\sqrt{2}} \langle (1,0) | (0,1) \rangle$$

$$= \frac{1}{\sqrt{2}}(2-i)$$

$$\Rightarrow \tilde{b}_2 = (0,1) - \frac{1}{\sqrt{2}}(2-i)\frac{1}{\sqrt{2}}(1,0)$$

$$= (0,1) - (1 - \tfrac{i}{2})(1,0) = (-1 + \tfrac{i}{2}, 1)$$

$$\langle \tilde{b}_2 | \tilde{b}_2 \rangle = 2\left[|-1 + \tfrac{i}{2}|^2 + 1 - i\left(-1 - \tfrac{i}{2} - (-1 + \tfrac{i}{2})\right)\right]$$

$$= 2\left[1 + \frac{1}{4} + \underbrace{1 - i(-i)}_{0}\right] = \frac{5}{2}$$

let $\boxed{b_2 := \frac{\tilde{b}_2}{\sqrt{\langle \tilde{b}_2 | \tilde{b}_2 \rangle}}} = \boxed{\sqrt{\frac{2}{5}}\,(-1 + \tfrac{i}{2}, 1)}$

Problem 3 Let V be a two-dimensional complex inner product space with inner product $\langle \cdot | \cdot \rangle$, $(|e_1\rangle, |e_2\rangle)$ be an orthonormal basis of V, and

$$|f_1\rangle := \frac{1}{\sqrt{2}}(|e_1\rangle + i|e_2\rangle), \qquad |f_2\rangle := \frac{1}{\sqrt{2}}(i|e_1\rangle + |e_2\rangle).$$

3.a (4 points) Show that $(|f_1\rangle, |f_2\rangle)$ is an orthonormal basis of V.

$$\langle f_1|f_1\rangle = \frac{1}{2}\left(\langle e_1| - i\langle e_2|\right)\left(|e_1\rangle + i|e_2\rangle\right) = \frac{1}{2}(1+1) = 1$$

$$\langle f_1|f_2\rangle = \frac{1}{2}\left(\langle e_1| - i\langle e_2|\right)\left(i|e_1\rangle + |e_2\rangle\right) = \frac{1}{2}(i-i) = 0$$

$$\langle f_2|f_1\rangle = \langle f_1|f_2\rangle^* = 0$$

$$\langle f_2|f_2\rangle = \frac{1}{2}\left(-i\langle e_1| + \langle e_2|\right)\left(i|e_1\rangle + |e_2\rangle\right) = \frac{1}{2}(1+1) = 1$$

$$\Rightarrow \langle f_i|f_j\rangle = \delta_{ij} \quad \forall i,j \in \{1,2\}$$

3.b (6 points) Let $U : V \to V$ be a unitary operator such that $U|e_1\rangle = |f_1\rangle$. Find all possible complex numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $U|e_2\rangle = \alpha_1|f_1\rangle + \alpha_2|f_2\rangle$.

$$\langle f_1|U|e_2\rangle = \alpha_1$$

Also $U|e_1\rangle = |f_1\rangle \Rightarrow |e_1\rangle = U^{-1}|f_1\rangle = U^\dagger|f_1\rangle$ $\Rightarrow$ $\boxed{\alpha_1 = 0}$

$$\Rightarrow \langle f_1|Ue_2\rangle = \langle U^\dagger f_1|e_2\rangle = \langle e_1|e_2\rangle = 0$$

$$\Rightarrow U|e_2\rangle = \alpha_2|f_2\rangle \Rightarrow \underset{\langle e_2|e_2\rangle = 1}{\langle Ue_2|Ue_2\rangle} = |\alpha_2|^2 \underbrace{\langle f_2|f_2\rangle}_{1}$$

$$\Rightarrow |\alpha_2| = 1 \Rightarrow \boxed{\alpha_2 = e^{i\varphi}} \text{ when } \varphi \in \mathbb{R}.$$

3.c (5 points) Find the matrix representation of U in the basis $(|e_1\rangle, |e_2\rangle)$.

$$\langle e_1|U|e_1\rangle = \langle e_1|f_1\rangle = \langle e_1|\left(\frac{1}{\sqrt{2}}(|e_1\rangle + i|e_2\rangle)\right) = \frac{1}{\sqrt{2}}$$

$$\langle e_1|U|e_2\rangle = \langle e_1|e^{i\varphi} f_2\rangle = e^{i\varphi}\langle e_1|f_2\rangle = e^{i\varphi}\frac{1}{\sqrt{2}}\langle e_1|\left(i|e_1\rangle + |e_2\rangle\right)$$

$$= \frac{i}{\sqrt{2}} e^{i\varphi}$$

$$\langle e_2|U|e_1\rangle = \langle e_2|f_1\rangle = \langle e_2|\left(\frac{1}{\sqrt{2}}|e_1\rangle + i|e_2\rangle\right) = \frac{i}{\sqrt{2}}$$

$$\langle e_2|U|e_2\rangle = \langle e_2|e^{i\varphi} f_2\rangle = e^{i\varphi}\langle e_2|\left(\frac{i(|e_1\rangle + |e_2\rangle)}{\sqrt{2}}\right)$$

$$= \frac{e^{i\varphi}}{\sqrt{2}}$$

$$\Rightarrow \mathbb{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}e^{i\varphi} \\ \frac{i}{\sqrt{2}} & \frac{e^{i\varphi}}{\sqrt{2}} \end{bmatrix}$$

Problem 4 (15 points) Let V be a finite-dimensional inner-product space, and A, B, and C be Hermitian operators acting in V and having V as their domain. Suppose that the eigenvalues of A are simple (nondegenerate) and $[A, B] = 0$ and $[A, C] = 0$. Show that $[B, C] = 0$.

Let $(e_1, \dots, e_n)$ be a basis of V consisting of eigenvectors of A.

$\forall i \in \{1, \dots, n\}$, $\quad A e_i = \alpha_i e_i$ for some $\alpha_i \in \mathbb{R}$

$$\Rightarrow \quad BAe_i = \alpha_i Be_i \quad \Rightarrow \quad A(Be_i) = \alpha_i (Be_i)$$
$$\qquad\quad \| $$
$$\qquad ABe_i = \alpha_i Be_i$$

$\Rightarrow$ $Be_i = 0$ or Be_i is an eigenvector of A with eigenvalue α_i

$\Downarrow$ e_i is an eigenvector of B

$\Downarrow$ $\exists \beta_i \in \mathbb{C}$, $Be_i = \beta_i e_i$

$\Downarrow$ e_i is an eigenvector of B

e_i's are eigenvectors of B. (1)

By the same argument e_i's are also eigenvectors of C, because $[A, C] = 0$. (2)

$\forall v \in V$, $\exists v_i \in \mathbb{C}$, $v = \sum_{i=1}^{n} v_i e_i \Rightarrow$

$$[B, C]v = (BC - CB) \sum_{i=1}^{n} v_i e_i$$
$$= B\Big(\sum_{i=1}^{n} v_i \underbrace{Ce_i}_{\gamma_i e_i}\Big) - C\Big(\sum_{i=1}^{n} v_i \underbrace{Be_i}_{\beta_i e_i}\Big)$$
$$= \sum_{i=1}^{n} v_i \gamma_i \underbrace{Be_i}_{\beta_i e_i} - \sum_{i=1}^{n} v_i \beta_i \underbrace{Ce_i}_{\gamma_i e_i}$$
$$= \sum_{i=1}^{n} (v_i \gamma_i \beta_i - v_i \beta_i v_i) = 0$$

$\Rightarrow [B, C] = 0$. ▨

Problem 5 Let $\mathscr{H}$ be a three-dimensional inner-product space, and $A : \mathscr{H} \to \mathscr{H}$ be a Hermitian operator that is represented in an orthonormal basis $\mathscr{E}$ of V by the matrix:

$$\mathbf{A} := \begin{bmatrix} 1 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

5.a (5 points) Find eigenvalues of A. These are the same as eigenvalues of $\mathbb{A}$.

$$\underbrace{\begin{bmatrix} 1-\lambda & i & 0 \\ -i & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}}_{M} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \det M = 0 \Rightarrow (1-\lambda)^3 - 2(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 2] = 0 \Rightarrow \begin{cases} \lambda = 1 \\ \text{or} \\ 1-\lambda = \pm\sqrt{2} \Rightarrow \lambda = 1 \mp \sqrt{2} \end{cases}$$

So eigenvalues of A are $\boxed{\lambda_1 = 1, \ \lambda_2 = 1+\sqrt{2}, \ \lambda_3 = 1-\sqrt{2}}$

5.b (15 points) Find the matrix representation Π_j in the basis $\mathscr{E}$ of the projection operators $\Pi_j : \mathscr{H} \to \mathscr{H}$ that map elements of $\mathscr{H}$ onto eigenvectors of A.

First we compute eigenvectors of $\mathbb{A}$:

$\lambda = \lambda_1 = 1 \Rightarrow \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} ib_1 = 0 \Rightarrow b_1 = 0 \\ -ia_1 + c_1 = 0 \Rightarrow c_1 = ia_1 \end{matrix}$

$\Rightarrow \mathcal{H}_1 = a_1 \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$ choose $a_1 = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\mathcal{H}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}}$

$\lambda = \lambda_2 = 1+\sqrt{2} \Rightarrow \begin{bmatrix} -\sqrt{2} & i & 0 \\ -i & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -\sqrt{2}a_2 + ib_2 = 0 \\ -ia_2 - \sqrt{2}b_2 + c_2 = 0 \\ b_2 - \sqrt{2}c_2 = 0 \end{matrix}$

$\Rightarrow a_2 = \frac{i}{\sqrt{2}} b_2$, $c_2 = \frac{b_2}{\sqrt{2}}$ $\Rightarrow \mathcal{H}_2 = b_2 \begin{bmatrix} \frac{i}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ choose $b_2 = \frac{1}{\sqrt{\frac{1}{2}+1+\frac{1}{2}}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{\mathcal{H}_2 = \frac{1}{2} \begin{bmatrix} i \\ \sqrt{2} \\ 1 \end{bmatrix}}$

$\lambda = \lambda_3 = 1-\sqrt{2} \Rightarrow \mathcal{H}_3 = b_3 \begin{bmatrix} -\frac{i}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ choose $b_3 = \frac{-1}{\sqrt{2}}$

$\Rightarrow \boxed{\mathcal{H}_3 = \frac{1}{2} \begin{bmatrix} i \\ -\sqrt{2} \\ 1 \end{bmatrix}}$

↓

5.b Continues:

$$\Pi_1 = \psi_1 \psi_1^\dagger = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}\begin{bmatrix} 1 & 0 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$$\Pi_2 = \psi_2 \psi_2^\dagger = \frac{1}{4}\begin{bmatrix} i \\ \sqrt{2} \\ 1 \end{bmatrix}\begin{bmatrix} -i & \sqrt{2} & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & i\sqrt{2} & i \\ -i\sqrt{2} & 2 & \sqrt{2} \\ -i & \sqrt{2} & 1 \end{bmatrix}$$

$$\Pi_3 = \psi_3 \psi_3^\dagger = \frac{1}{4}\begin{bmatrix} i \\ -\sqrt{2} \\ 1 \end{bmatrix}\begin{bmatrix} -i & -\sqrt{2} & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -i\sqrt{2} & i \\ i\sqrt{2} & 2 & -\sqrt{2} \\ -i & -\sqrt{2} & 1 \end{bmatrix}$$

Problem 6 Let $\mathscr{H}$ and A of Problem 5 be the Hilbert space and an observable of a quantum system, and λ_ψ be the state given by a state vector ψ that we can represent in the basis $\mathscr{E}$ by $\psi = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate the probability of finding the following values as a result of measuring A in the state λ_ψ.

6.a (8 points) $\alpha = 1$.

$$\mathcal{P}_1(\lambda_\psi) = \frac{\langle\psi|\Pi_1|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\psi^\dagger \Pi_1 \psi}{\psi^\dagger\psi}$$

$$\psi^\dagger \Pi_1 \psi = [1 \;\; 1 \;\; 1]\frac{1}{2}\begin{bmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}[1 \;\; 1 \;\; 1]\begin{bmatrix} 1-i \\ 0 \\ 1+i \end{bmatrix} = \frac{1}{2}(1-i+1+i) = 1$$

$$\psi^\dagger\psi = [1 \;\; 1 \;\; 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

$$\Rightarrow \boxed{\mathcal{P}_1(\lambda_\psi) = \frac{1}{3}}$$

6.b (7 points) $\alpha = 1 + \sqrt{2}$.

$$\mathcal{P}_{1+\sqrt{2}}(\lambda_\psi) = \frac{\langle\psi|\Pi_2|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\psi^\dagger \Pi_2 \psi}{\psi^\dagger\psi}$$

$$\psi^\dagger \Pi_2 \psi = [1 \;\; 1 \;\; 1]\frac{1}{4}\begin{bmatrix} 1 & i\sqrt{2} & i \\ -i\sqrt{2} & 2 & \sqrt{2} \\ -i & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4}[1 \;\; 1 \;\; 1]\begin{bmatrix} 1+(1+\sqrt{2})i \\ 2+\sqrt{2}-i\sqrt{2} \\ 1+\sqrt{2}-i \end{bmatrix} = \frac{1}{4}(1+2+\sqrt{2}+1+\sqrt{2})$$

$$= 1 + \frac{\sqrt{2}}{2}$$

$$\Rightarrow \boxed{\mathcal{P}_{1+\sqrt{2}}(\lambda_\psi) = \frac{2+\sqrt{2}}{6}}$$

6.c (5 points) $\alpha = 2 + \sqrt{2}$.

$2+\sqrt{2} \notin$ Spectrum of A $\Rightarrow \boxed{\mathcal{P}_{2+\sqrt{2}}(\lambda_\psi) = 0}$