Phys 401: Midterm Exam 1

Fall 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deduction of you grade.

Problem 1 Consider a point particle of mass m that moves on a straight line under the influence of a conservative force $F = F(x)$ where x denotes the position of the particle.

1.a (10 points) Prove that the total energy of the system is conserved.

According to Newton's eqn: $m\ddot{x}(t) = F(x(t))$

$E = \frac{m\dot{x}^2}{2} + V(x)$ where $V(x)$ is a potential

such that $F = -V'$

$$\frac{d}{dt}E = m\dot{x}\ddot{x} + \frac{dV}{dt} = m\dot{x}\ddot{x} + V'\dot{x}$$

$$= \dot{x}(m\ddot{x} + V') = \dot{x}(m\ddot{x} - F) = 0$$

$\Rightarrow E = \text{constant}.$

Now, let F be given by $F(x) := \alpha e^{-x/\ell}$, where α and ℓ are nonzero real constants.

1.b (10 points) Give the Euler-Lagrange equation for the system, simplify it as much as possible, and show that it is equivalent to Newton's equation for this system.

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - V(x)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -V', \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \implies m\ddot{x} - (-V') = 0$$

$$\implies m\ddot{x} = -V' = F(x) = \alpha e^{-\frac{x}{\ell}}$$

for $F = \alpha e^{-\frac{x}{\ell}}$ $\quad V = -\int F\,dx = -\left(\alpha \frac{e^{-\frac{x}{\ell}}}{-\frac{1}{\ell}} + c\right) = \alpha \ell e^{-\frac{x}{\ell}} - c$ ✓

we can set $c = 0$ without loss of generality, so that $V(x) = \alpha \ell e^{-\frac{x}{\ell}}$ const.

1.c (10 points) Give Hamilton's equations for the system, simplify them as much as possible, and show that they are equivalent to the Newton's equation.

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \implies \dot{x} = \frac{p}{m}$$

$$H = \dot{x}p - \mathcal{L} = \frac{p^2}{m} - \left[\frac{1}{2m}\left(\frac{p}{m}\right)^2 - V(x)\right]$$

$$= \frac{p^2}{2m} + V(x) \qquad V(x) = \alpha \ell e^{-\frac{x}{\ell}}$$

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \implies p = m\dot{x} \\ \dot{p} = -\frac{\partial H}{\partial x} = -V' \end{cases}$$

$$\hookrightarrow \frac{d}{dt}(m\dot{x}) = -V'$$

$$m\ddot{x} = F(x) \quad ✓$$

$$\implies m\ddot{x} = \alpha e^{-\frac{x}{\ell}}$$

Problem 2 (20 points) In Problem 1, take $x(0) = 0$ and $\dot{x}(0) = 0$, and use conservation of total energy to obtain the position of the particle for $t > 0$. To get full credit you must simplify the expression you find for $x(t)$ as much as possible.
Hint: For any real number b,

$$\int \frac{dx}{\sqrt{b^2 - e^{-x/\ell}}} = \left(\frac{\ell}{b}\right) \ln\left(\frac{1+\sqrt{1-b^{-2}e^{-x/\ell}}}{1-\sqrt{1-b^{-2}e^{-x/\ell}}}\right) + c,$$

where c is an integration constant.

$$E = \frac{m\dot{x}^2}{2} + V(x) = \frac{m\dot{x}^2}{2} + \alpha \ell e^{-\frac{x}{\ell}}$$

$$\Rightarrow \quad \dot{x} = \pm\sqrt{\frac{2}{m}\left(E - \alpha\ell e^{-\frac{x}{\ell}}\right)}$$

Becaus $\quad \dot{x}(0) = 0 \;,\; x(0) = 0 \Rightarrow \quad E = \alpha\ell \quad \Rightarrow$

$$\dot{x} = \pm\sqrt{\frac{2\alpha\ell}{m}}\sqrt{1-e^{-\frac{x}{\ell}}} \quad \text{for } \dot{x}\in\mathbb{R} \Rightarrow \boxed{\alpha\ell > 0}$$

$$\Rightarrow \int_0^x \frac{dx'}{\sqrt{1-e^{-\frac{x'}{\ell}}}} = \pm\sqrt{\frac{2\alpha\ell}{m}}(t - \underset{0}{\underset{\parallel}{t_0}}) = \pm\sqrt{\frac{2\alpha\ell}{m}}\, t$$

$$\parallel$$

$$\ell \ln\left(\frac{1+\sqrt{1-e^{-x/\ell}}}{1-\sqrt{1-e^{-x/\ell}}}\right)\Bigg|_0^x = \pm\sqrt{\frac{2\alpha\ell}{m}}\, t$$

$$\Rightarrow \ell \ln\left(\frac{1+\sqrt{1-e^{-x/\ell}}}{1-\sqrt{1-e^{-x/\ell}}}\right) = \pm\sqrt{\frac{2\alpha\ell}{m}}\, t$$

$$t>0 \Rightarrow \pm = \text{Sign}\,\ell = \text{Sign}\,\alpha \quad \hookrightarrow \ln\left(\frac{1+\sqrt{1-e^{-x/\ell}}}{1-\sqrt{1-e^{-x/\ell}}}\right) = \underbrace{\frac{1}{|\ell|}\sqrt{\frac{2\alpha\ell}{m}}}\, t$$

let $\boxed{\beta := \sqrt{\frac{2\alpha}{m\ell}}}$

$$\Rightarrow \frac{1+\sqrt{1-e^{-x/\ell}}}{1-\sqrt{1-e^{-x/\ell}}} = \frac{e^{\beta t}}{1}$$

$$\Rightarrow \frac{1}{2}\left(1+\sqrt{1-e^{-x/\ell}}\right) = \frac{e^{\beta t}}{1+e^{\beta t}} \Rightarrow 1-e^{-\frac{x}{\ell}} = \left(\frac{2e^{\beta t}}{1+e^{\beta t}} - 1\right)^2 = \left(\frac{e^{\beta t}-1}{1+e^{\beta t}}\right)^2$$

$$\Rightarrow x = -\ell \ln\left[1 - \left(\frac{e^{\beta t}-1}{e^{\beta t}+1}\right)^2\right] = -\ell \ln\left[\left(\frac{2}{e^{\beta t}+1}\right)\left(\frac{2e^{\beta t}}{e^{\beta t}+1}\right)\right]$$

$$= \ell\left[2\ln\left(\frac{e^{\beta t}+1}{2}\right) - \beta t\right]$$

Problem 3 (20 points) Give the Hamilton-Jacobi equation for the system described in Problem 1. Show that the solution of this equation reproduces your response to Problem 2.

$$-\frac{\partial}{\partial t} S = H(x, S') = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x), \quad S' := \frac{\partial S}{\partial x}$$

let $S = -tE + W(x) \Rightarrow \quad p = \frac{\partial S}{\partial x} = W' := \frac{\partial W}{\partial x}$

$$\Rightarrow E = \frac{W'^2}{2m} + \alpha l e^{-\frac{x}{l}}$$

let $\tilde{x} := x(0)$, $\tilde{p} := p(0) \Rightarrow \boxed{E = \frac{\tilde{p}^2}{2m} + \alpha l e^{-\frac{\tilde{x}}{l}}}$ (*)

Also we have $\boxed{W = \pm \int dx \sqrt{2m(E - \alpha l e^{-\frac{x}{l}})}}$

$$\tilde{p} = -\frac{\partial S}{\partial \tilde{x}} = -t\frac{\partial E}{\partial \tilde{x}} - \frac{\partial W}{\partial \tilde{x}} = \left(-t - \frac{\partial W}{\partial E}\right)\frac{\partial E}{\partial \tilde{x}}$$

$$\frac{\partial E}{\partial \tilde{x}} = -\alpha e^{-\frac{\tilde{x}}{l}} = -\alpha \neq 0$$

$\tilde{x} = x(0) = 0$

$\tilde{p} = p(0) = 0$

$$0 = \left(-t - \frac{\partial W}{\partial E}\right)\underbrace{\frac{\partial E}{\partial \tilde{x}}}_{-\alpha} \Rightarrow \boxed{\frac{\partial W}{\partial E} = -t}$$

$$\pm \int dx \frac{m}{\sqrt{2m(E - \alpha l e^{-\frac{x}{l}})}} = -t$$

$$\Rightarrow \quad \pm \frac{m}{\sqrt{2mE}} \int \frac{dx}{\sqrt{1 - \frac{\alpha l}{E} e^{-\frac{x}{l}}}} = -t$$

From (*) $E = \alpha l \quad \hookrightarrow \quad \boxed{\int \frac{dx}{\sqrt{1 - e^{-\frac{x}{l}}}} = \pm \sqrt{\frac{2\alpha l}{m}}\, t}$

This is precisely the eqn we derived in Problem 2. So it gives the same solution.

Problem 4 Consider a particle moving on the real line whose motion is described by the Hamiltonian

$$H = \alpha\, e^{\mu t} q^2 p^{4/3} + \beta\, e^{-\mu t} p^{2/3},$$

where α, β, and μ are positive real parameters, and q and p are respectively the position and momentum of the particle.

4.a (5 points) Find a function $f(t)$ such that for every real number ν the following coordinate transformation of the phase space of this particle is a canonical transformation: $(q, p) \to (\tilde{q}, \tilde{p})$ where

$$\tilde{q} := e^{\nu t} p^{1/3}, \qquad \tilde{p} := f(t)\, q\, p^{2/3}.$$

$$1 = \{\tilde q, \tilde p\} = \underbrace{\frac{\partial \tilde q}{\partial q}}_{0}\frac{\partial \tilde p}{\partial p} - \frac{\partial \tilde q}{\partial p}\frac{\partial \tilde p}{\partial q}$$

$$= -\left(\frac{1}{3} e^{\nu t} p^{-2/3}\right)\left(f(t)\, p^{2/3}\right) = -\frac{e^{\nu t} f(t)}{3}$$

$$\Rightarrow \boxed{f(t) = -3 e^{-\nu t}}$$

4.b (5 points) Choose ν so that when you express the right-hand side of the above expression for H in terms of $\tilde q$ and $\tilde p$ it does not depend on t explicitly.

$$H = \alpha e^{\mu t} q^2 p^{4/3} + \beta e^{-\mu t} p^{2/3} = \alpha e^{\mu t}\left(\frac{\tilde p}{f(t)}\right)^2 + \beta e^{-\mu t}\left(\frac{\tilde q}{e^{\nu t}}\right)^2$$

$$= \alpha e^{\mu t}\left(\frac{\tilde p^2}{9 e^{-2\nu t}}\right) + \beta e^{-\mu t}\left(\frac{\tilde q^2}{e^{2\nu t}}\right)$$

So take $\mu = -2\nu \Rightarrow \boxed{\nu = -\frac{\mu}{2}}$

$$\Rightarrow H = \frac{\alpha \tilde p^2}{9} + \beta \tilde q^2$$

4.c (15 points) For the function $f(t)$ and the value of ν you find in Problems 4.a and 4.b, $(q,p) \to (\tilde{q},\tilde{p})$ is a canonical transformation. Find a generator of the form $F(q,\tilde{q})$ for this canonical transformation.

Hint: Recall that

$$p = \frac{\partial F}{\partial q}, \qquad \tilde{p} = -\frac{\partial F}{\partial \tilde{q}}, \qquad \tilde{H} = H + \frac{\partial F}{\partial t}.$$

$$F = \int p\,dq + g(\tilde{q}) \qquad p = (e^{-\nu t}\tilde{q})^3 = (e^{\frac{\mu}{2}t}\tilde{q})^3 = e^{\frac{3\mu}{2}t}\tilde{q}^3$$

$$\Rightarrow \quad F = \int e^{\frac{3\mu}{2}t}\tilde{q}^3\,dq + g(\tilde{q}) = e^{\frac{3\mu}{2}t}\tilde{q}^3 q + g(\tilde{q})$$

$$\tilde{p} = -\frac{\partial F}{\partial \tilde{q}} \Rightarrow \underbrace{f(t)}_{-3e^{-\nu t} = -3e^{\frac{\mu}{2}t}}\, q\, \underbrace{p^{2/3}}_{(e^{\frac{3\mu}{2}t}\tilde{q}^3)^{2/3} = e^{\mu t}\tilde{q}^2} = -3e^{\frac{3\mu}{2}t}\tilde{q}^2 q - g'(\tilde{q})$$

$$\Rightarrow \quad -3e^{\frac{3\mu}{2}t} q\tilde{q}^2 = -3e^{\frac{3\mu}{2}t}\tilde{q}^2 q - g'(\tilde{q})$$

$$\Rightarrow \quad g'(\tilde{q}) = 0 \quad \Rightarrow \quad g(\tilde{q}) = \text{const} =: g_0$$

$$\Downarrow$$

$$\boxed{F(q,\tilde{q}) = e^{\frac{3\mu}{2}t} q\tilde{q}^3 + g_0}$$

4.d (5 points) Determine the transformed Hamiltonian $\tilde{H}$ and simplify it as much as possible.

$$\tilde{H} = H + \frac{\partial F}{\partial t}$$

$$\frac{\partial F}{\partial t} = \frac{3}{2}\mu e^{\frac{3\mu}{2}t} q \tilde{q}^3$$

$$q = \frac{\tilde{p}}{f(t)} p^{-2/3} = \left(\frac{\tilde{p}}{-3e^{\frac{\mu t}{2}}}\right)\left(\frac{\tilde{q}}{e^{-\frac{\mu}{2}t}}\right)^{-2}$$

$$= \frac{\tilde{p}}{-3e^{\frac{3\mu t}{2}}\tilde{q}^2}$$

$$\Rightarrow \frac{\partial F}{\partial t} = -\frac{1}{2}\mu \tilde{q}\tilde{p}$$

$$\Rightarrow \boxed{\tilde{H} = \frac{\alpha \tilde{p}^2}{g} + \beta \tilde{q}^2 - \frac{1}{2}\mu \tilde{q}\tilde{p}}$$