Phys 401/OEPE 541: Final Exam

January 05, 2018

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 3 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.

Problem 1 (15 points) Consider a classical particle moving in $\mathbb{R}$. Let X and P be the corresponding position and momentum observables and $\tilde{X} := e^{\alpha P} f(X)$ and $\tilde{P} := e^{\beta X + \gamma P}$ where α, β, γ are real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Find necessary and sufficient conditions on α, β, γ, and f such that $(X, P) \to (\tilde{X}, \tilde{P})$ is a canonical transformation.

$$1 = \{\tilde{X}, \tilde{P}\}_{PB} = \frac{\partial}{\partial x}(e^{\alpha P} f)\frac{\partial}{\partial p}(e^{\beta x + \gamma P}) - \frac{\partial}{\partial p}(e^{\alpha P} f)\frac{\partial}{\partial x}(e^{\beta x + \gamma P})$$

$$= e^{\alpha P} f' e^{\beta x} \gamma e^{\gamma P} - \alpha e^{\alpha P} f \beta e^{\beta x + \gamma P}$$

$$= e^{(\alpha+\gamma)P} e^{\beta x} (\gamma f' - \alpha\beta) f$$

$$\Rightarrow \underbrace{e^{\beta x}(\gamma f' - \alpha\beta f)}_{x\text{-dep.}} = \underbrace{e^{-(\alpha+\gamma)P}}_{p\text{-dep}} = \text{const} \Rightarrow \alpha + \gamma = 0 \Rightarrow \boxed{\gamma = -\alpha}$$

$$\Downarrow$$

$$\text{Const} = 1$$

$$\Rightarrow \underset{-\alpha}{\underset{\shortparallel}{\gamma}} f' - \alpha\beta f = e^{-\beta x} \Rightarrow \begin{cases} \alpha \neq 0 \\ \text{and} \\ f' + \beta f = -\dfrac{e^{-\beta x}}{\alpha} \end{cases}$$

$$f' + \beta f = 0 \Rightarrow f = c e^{-\beta x} \quad \text{try} \quad f = \phi e^{-\beta x} \Rightarrow$$

$$(\phi' + \beta\phi) e^{-\beta x} + \beta e^{-\beta x} \phi = -\frac{e^{-\beta x}}{\alpha} \Rightarrow \phi' = -\frac{1}{\alpha} \Rightarrow \phi = -\frac{x}{\alpha} + c$$

$$\Rightarrow \boxed{f(x) = (c - \tfrac{x}{\alpha}) e^{-\beta x}}$$

So the neccessary & sufficient condition is

$\gamma = -\alpha \neq 0$ and $f(x) = (c - \frac{x}{\alpha}) e^{-\beta x}$ for some $c \in \mathbb{R}$.

Problem 1 (20 points) Determine the general form of the functions $f:\mathbb{R}\to\mathbb{C}$, $g:\mathbb{R}\to\mathbb{C}$, and $h:\mathbb{R}\to\mathbb{C}$ such that

$$\psi(x,t) = f(t)e^{-g(t)x^2+h(t)x}$$

is the position wave function for an evolving state vector $|\psi(t)\rangle$ of a free particle of mass m moving in $\mathbb{R}$.

Note: $|\psi(t)\rangle$ need not be normalized.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t} \qquad (I)$$

$$\frac{\partial\psi}{\partial x} = f\,(-2gx+h)\,e^{-gx^2+hx}$$

$$\frac{\partial^2\psi}{\partial x^2} = f\,[-2g+(-2gx+h)^2]\,e^{-gx^2+hx}$$

$$= f\,(4g^2x^2-4gxh+h^2-2g)\,e^{-gx^2+hx}$$

$$\frac{\partial\psi}{\partial t} = [f' + (-g'x^2+h'x)f\,]\,e^{-gx^2+ht}$$

$$(I) \Leftrightarrow -\frac{\hbar^2}{2m}\left[4g^2x^2-4gxh+h^2-2g\right] = i\hbar\left[\frac{f'}{f}+(-g'x^2+h'x)\right]$$

$$\Leftrightarrow \begin{cases} \frac{i\hbar}{2m}(4g^2) = -g' \Rightarrow -\frac{g'}{g^2} = \frac{2i\hbar}{m} \Rightarrow \frac{1}{g} = \frac{2i\hbar}{m}(t-t_0) \Rightarrow \boxed{g = \frac{m}{2i\hbar(t-t_0)}} \\ \frac{i\hbar}{2m}(-4gh) = h' \longrightarrow \frac{h'}{h} = -\frac{2i\hbar g}{m} = -\frac{1}{t-t_0} \\ \frac{i\hbar}{2m}(h^2-2g) = \frac{f'}{f} \end{cases}$$

$$\Downarrow$$

$$\ln|h| = -\ln|t-t_0| + c_0$$

$$\Downarrow$$

$$\boxed{h = \frac{h_0}{|t-t_0|}} \qquad h_0 \in \mathbb{C}$$

$$\Rightarrow \frac{f'}{f} = \frac{i\hbar}{2m}\left[\frac{h_0^2}{(t-t_0)^2} - \frac{m}{i\hbar(t-t_0)}\right] = \frac{i\hbar h_0^2}{2m(t-t_0)^2} - \frac{1}{2(t-t_0)}$$

$$\Rightarrow \ln|f| = \frac{i\hbar h_0^2}{2m}\left(-\frac{1}{t-t_0}\right) - \frac{1}{2}\ln|t-t_0| + c_1$$

$$\Rightarrow \boxed{f = f_0\, e^{-\frac{i\hbar h_0^2}{2m(t-t_0)}}\,|t-t_0|^{-\frac{1}{2}}} \qquad f_0 \in \mathbb{C}$$

Problem 3 Consider a particle that is moving in $\mathbb{R}$ and interacting with a potential of the form

$$V(x) = \begin{cases} +\infty & \text{for} \quad x \leq 0, \\ -V_0 & \text{for} \quad x \in (0, b), \\ 0 & \text{for} \quad x \geq b, \end{cases}$$

where b and V_0 are positive real constants. Suppose that the particle is in a state given by the position wave function:

$$\psi(x) = \begin{cases} x\, e^{-x/b} & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$

3.a (10 points) Find the expectation value of the position of the particle.

Useful formula: $\displaystyle\int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ where $n \in \{0, 1, 2, \cdots\}$ and $\alpha \in \mathbb{R}^+$.

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_0^\infty x^2 e^{-\frac{2x}{b}} dx$$

$$= \frac{2!}{\left(\frac{2}{b}\right)^3} = \frac{b^3}{4}$$

$$\langle \psi | \hat{X} | \psi \rangle = \int_{-\infty}^{\infty} x\, |\psi(x)|^2 dx = \int_0^\infty x^3 e^{-\frac{2x}{b}} dx$$

$$= \frac{3!}{\left(\frac{2}{b}\right)^4} = \frac{3b^4}{8}$$

$$\Rightarrow \quad \langle \hat{X} \rangle = \frac{\langle \psi | \hat{X} | \psi \rangle}{\langle \psi | \psi \rangle} = \left(\frac{3b^4}{8}\right)\left(\frac{4}{b^3}\right) = \frac{3}{2} b$$

3.b (5 points) Find the uncertainty in the position of the particle.

$$\langle \psi | \hat{x}^2 | \psi \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \, dx = \int_{0}^{\infty} x^4 e^{-\frac{2x}{b}} \, dx$$

$$= \frac{4!}{\left(\frac{2}{b}\right)^5} = \frac{24 b^5}{32} = \frac{3}{4} b^5$$

$$\langle \hat{x}^2 \rangle = \frac{\langle \psi | \hat{x}^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \left(\frac{3}{4} b^5\right)\left(\frac{4}{b^3}\right) = 3 b^2$$

$$\Delta \hat{x} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$$

$$= \sqrt{3 b^2 - \left(\frac{3}{2} b\right)^2} = \sqrt{3 - \frac{9}{4}} \, b$$

$$= \frac{\sqrt{3}}{2} b$$

3.c (10 points) Find the expectation value of the energy of the particle.

Useful formula: $\int x^2 e^{-\alpha x} dx = -\frac{[(\alpha x+1)^2+1]\, e^{-\alpha x}}{\alpha^3} + c.$

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{x})$$

$$\langle \psi | \hat{P}^2 | \psi \rangle = \langle \hat{P}\psi | \hat{P}\psi \rangle = \| \hat{P}\psi \|^2 = \int_{-\infty}^{\infty} |\langle x | \hat{P}\psi \rangle|^2 dx$$

$$\langle x | \hat{P}\psi \rangle = -i\hbar \frac{d}{dx}\psi(x) = -i\hbar \frac{d}{dx}\left[x e^{-\frac{x}{b}} \theta(x) \right]$$

$$= -i\hbar \left[\left(1 - \frac{x}{b}\right) e^{-\frac{x}{b}} \theta(x) + \underbrace{x e^{-\frac{x}{b}} \delta(x)}_{0} \right]$$

$$= -i\hbar \left(1 - \frac{x}{b}\right) e^{-\frac{x}{b}} \theta(x) \qquad \theta(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\Rightarrow \boxed{\langle \psi | \hat{P}^2 | \psi \rangle} = \int_0^\infty \hbar^2 \left(1 - \frac{x}{b}\right)^2 e^{-\frac{2x}{b}} dx = \hbar^2 \int_0^\infty \left(1 - \frac{2x}{b} + \frac{x^2}{b^2}\right) e^{-\frac{2x}{b}} dx$$

$$= \hbar^2 \left[\underbrace{\int_0^\infty e^{-\frac{2x}{b}} dx}_{\frac{1}{\left(\frac{2}{b}\right)}} - \frac{2}{b} \underbrace{\int_0^\infty x e^{-\frac{2x}{b}} dx}_{\frac{1}{\left(\frac{2}{b}\right)^2}} + \frac{1}{b^2} \underbrace{\int_0^\infty x^2 e^{-\frac{2x}{b}} dx}_{\frac{2}{\left(\frac{2}{b}\right)^3}} \right]$$

$$= \hbar^2 \left(\frac{b}{2} - \frac{b}{2} + \frac{b}{4} \right) = \boxed{\frac{\hbar^2 b}{4}}$$

$$\boxed{\langle \psi | V(\hat{x}) | \psi \rangle} = \int_{-\infty}^{\infty} V(x) |\psi(x)|^2 dx = -V_0 \int_0^b x^2 e^{-\frac{2x}{b}} dx$$

$$= -V_0 \left\{ -\left(\frac{2}{b}\right)^{-3} \left[\left(\frac{2}{b}x + 1\right)^2 + 1 \right] e^{-\frac{2x}{b}} \Big|_0^b \right\}$$

$$= \frac{V_0 b^3}{8} \left\{ \underbrace{[(2+1)^2 + 1]}_{10} e^{-2} - \underbrace{[1+1]}_{2} \right\} = \boxed{\frac{V_0 b^3}{4} \left(5e^{-2} - 1 \right)}$$

$$\langle \psi | \hat{H} | \psi \rangle = \frac{1}{2m} \langle \psi | \hat{P}^2 | \psi \rangle + \langle \psi | V(\hat{x}) | \psi \rangle = \frac{\hbar^2 b}{8m} + \frac{V_0 b^3}{4} \left(\frac{5}{e^2} - 1 \right)$$

$$\Rightarrow \langle \hat{H} \rangle = \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} = \left(\frac{4}{b^3} \right) \left[\frac{\hbar^2 b}{8m} + \frac{V_0 b^3}{4} \left(\frac{5}{e^2} - 1 \right) \right]$$

$$\Rightarrow \boxed{\langle \hat{H} \rangle = \frac{\hbar^2}{2mb^2} + V_0 \left(\frac{5}{e^2} - 1 \right)}$$

Problem4 The motion of a free particle of mass m that is constrained to a circle of radius a may be identified with that of a free particle moving in the interval $[-\pi a, \pi a]$ and having its position wave functions $\psi(x)$ satisfy the periodic boundary conditions:

$$\psi(-\pi a) = \psi(\pi a) \qquad \psi'(-\pi a) = \psi'(\pi a),$$

where a prime stands for differentiation with respect to x.

4.a (15 points) Find the energy spectrum of such a particle, i.e., solve the time-independent Schrödinger equation for a free particle in the interval $[-\pi a, \pi a]$ and impose the above periodic boundary conditions. What is the degree of degeneracy of energy eigenvalues?

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \qquad \text{let} \quad k := \frac{\sqrt{2mE}}{\hbar} \quad \text{so that}$$

$$\psi'' + k^2\psi = 0$$

$$E = \frac{\langle\psi| -\frac{\hbar^2}{2m}\psi''\rangle}{\langle\psi|\psi\rangle} = \frac{\langle\psi|\hat{P}^2\psi\rangle}{\langle\psi|\psi\rangle} \geq 0 \quad \Rightarrow \quad k \in \mathbb{R}^+ \cup \{0\}$$

$$\Rightarrow \quad \psi = c\cos(kx) + d\sin(kx)$$

$$\psi(-\pi a) = c\cos(-\pi ak) + d\sin(-\pi ak) = c\cos(\pi ak) - d\sin(\pi ak)$$

$$\psi(\pi a) \Rightarrow c\cos(\pi ak) + d\sin(\pi ak) \quad \hookrightarrow \quad \boxed{d\sin(\pi ak) = 0}$$

$$\psi'(x) = k[-c\sin(kx) + d\cos(kx)]$$

$$\psi'(-\pi a) = k[c\sin(\pi ak) + d\cos(\pi ak)]$$

$$\psi'(\pi a) \Rightarrow \boxed{kc\sin(\pi ak) = 0}$$

So either $c = d = 0 \Rightarrow \psi = 0 \Rightarrow$ ※

or $\sin(\pi ak) = 0 \Rightarrow ak = n \quad n \in \mathbb{Z}$

$$\Rightarrow \quad k = \frac{\pi n}{a} \qquad n \in \{0, \pm 1, \pm 2, \cdots\}$$

$$\Rightarrow \quad E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2} \qquad n \in \{0, 1, 2, \cdots\} = \mathbb{N}$$

$$\text{Energy Spectrum} = \left\{ \frac{\hbar^2\pi^2 n^2}{2ma^2} \,\middle|\, n \in \mathbb{N} \right\}.$$

4.b (10 points) Find a complete orthonormal set of energy eigenfunctions for the system.

For each $n \in \{0, 1, 2, \dots\}$ $\exists$ two eigenfunctions

$$\psi_{n1}(x) := d_n \sin\left(\frac{\pi n x}{a}\right) \quad \& \quad \psi_{n2}(x) = c_n \cos\left(\frac{\pi n x}{a}\right)$$

$$\|\psi_{n1}\|^2 = 1 \Rightarrow |d_n|^2 \int_{-\pi a}^{\pi a} \underbrace{\sin^2\left(\frac{\pi n x}{a}\right)}_{\frac{1 - \cos\frac{2\pi n x}{a}}{2}} dx = 1 \Rightarrow |d_n|^2 \pi a = 1$$

$$\Downarrow \qquad |d_n| = \frac{1}{\sqrt{\pi a}}$$

Similar $\|\psi_{n2}\|^2 = 1 \Rightarrow |c_n| = \frac{1}{\sqrt{\pi a}}$

we can take $d_n = \frac{1}{\sqrt{\pi a}} = c_n \Rightarrow$

$$\psi_{n1} = \frac{1}{\sqrt{\pi a}} \sin\left(\frac{\pi n x}{a}\right), \quad \psi_{n2} = \frac{1}{\sqrt{\pi a}} \cos\left(\frac{\pi n x}{a}\right)$$

4.c (5 points) Let $\hat{\mathscr{P}}$ be the operator defined by $(\hat{\mathscr{P}}\psi)(x) = \psi(-x)$. Determine a complete set of energy eigenfunctions of the particle that are also the eigenfunctions of $\hat{\mathscr{P}}$.

$$(\hat{\mathscr{P}}\psi_{n1})(x) = \frac{1}{\sqrt{\pi a}} \sin\left(-\frac{\pi n x}{a}\right) = -\psi_{n1}(x)$$

$$(\hat{\mathscr{P}}\psi_{n2})(x) = \frac{1}{\sqrt{\pi a}} \cos\left(-\frac{\pi n x}{a}\right) = \psi_{n2}(x)$$

$$\Rightarrow \quad \hat{\mathscr{P}}|\psi_{n\epsilon}\rangle = (-1)^{\epsilon} |\psi_{n\epsilon}\rangle$$

So $\{|\psi_{n\epsilon}\rangle \mid n \in \mathbb{N}, \epsilon \in \{1, 2\}\}$ is such a basis.

4.d (10 points) Let α be a real variable, find the spectrum of the operator $\hat{H}_\alpha := \frac{\hat{P}^2}{2m} + \alpha\hat{\mathscr{P}}$.

$|\psi_{n\epsilon}\rangle$ are eigenvectors of both $\frac{\hat{P}^2}{2m}$ & $\hat{\mathscr{P}}$, they are also complete $\Rightarrow$ they form a basis of eigenvectors of $\hat{H}_\alpha$ $\Rightarrow$

$$\hat{H}_\alpha |\psi_{n\epsilon}\rangle = E_n |\psi_{n\epsilon}\rangle + \alpha(-1)^\epsilon |\psi_{n\epsilon}\rangle$$

$$= \left(E_n + (-1)^\epsilon \alpha\right) |\psi_{n\epsilon}\rangle$$

$\Rightarrow$ Spectrum of $\hat{H}_\alpha = \left\{ \frac{\hbar^2 \pi^2 n^2}{2ma^2} + (-1)^\epsilon \alpha \;\middle|\; n \in \mathbb{N},\ \epsilon \in \{1, 2\} \right\}$