Math 503: Midterm Exam 1

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have two hours.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1. Let V and W be a complex inner product spaces and $L : V \to W$ be a linear operator.

1.a) Show that the null space of L is a subspace of V. (5 points)

1.b) Show that the range of L is a subspace of W. (5 points)

1.c) Show that if $V = W$ and L is self-adjoint, then every element of the null-space of L is orthogonal to every element of its range. (5 points)

1.d) Show that if $V = W$, L is self-adjoint, and $L^2 = 0$, then $L = 0$. (5 points)

Problem 2. Let V be a finite dimensional complex inner product, B be an orthonormal basis of V, and $L : V \to V$ be a linear operator having V as it domain. Show that if the matrix representation of L in the basis B is a Hermitian matrix, then L must be self-adjoint. (15 points)

Problem 3. Let $\mathcal{C}(\mathbb{R})$ be the real vector space of functions $f : \mathbb{R} \to \mathbb{R}$ with domain $\mathbb{R}$, $D : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ and $I : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ be the derivative operator and the identity operator, respectively, $f, g \in \mathcal{C}(\mathbb{R})$ be nonzero differentiable functions, and $L := fD + gI$. Determine the most general first order differential operator J satisfying $LJ = JL$ and express J in terms of L. (15 points)

Problem 4. Given that $y_1(x) := e^x$ is a solution of $xy'' - (1+x)y' + y = 0$ find the general solution of the following equation for $x > 0$.

$$xy'' - (1+x)y' + y = x^2 e^x. \qquad \text{(20 points)}$$

Note: Try to check whether your solution actually solves the equation.

Problem 5. Consider the equation: $xy'' - y = 0$.

5.a) Find a nonzero power series solution of this equation that satisfies $y'(0) = 1$. (15 points)

5.b) Denote the solution you found in part 5.a of this problem by y_1. Use the ratio test to find the radius of convergence of y_1. (5 points)

5.c) Find a recurrence relation for b_n such that $y_2(x) := \ln(x)y_1(x) + \sum_{n=0}^{\infty} b_n x^n$ is a solution of $xy'' - y = 0$ for $x > 0$. (10 points)