Math 503, Fall 2007
Assignment for October 09-15

- Read pages 333-348 of the textbook (Kreyszig, 9th Edition)
- Solve Problems 6, 10, 14, 29, 30 on pages 338-339 of the textbook.
- Solve the following problems.

1. Let V and W be complex vector spaces and $L : V \to W$ be a linear operator.
   (a) Prove that the null space of L, i.e., $\text{null}(L) := \{v \in V \mid Lv = 0\}$, is a subspace of V.
   (b) Prove that the range of L, i.e., $\text{range}(L) := \{w \in W \mid \exists v \in V, w = Lv\}$, is a subspace of W.

2. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined by $\forall x \in \mathbb{R}$, $f_1(x) := \sin x$ and $f_2(x) := \cos x$, $V := \{a_1 f_1 + a_2 f_2 | a_1, a_2 \in \mathbb{R}\}$, and $D : V \to V$ denote the differentiation.
   (a) Prove that V is a subspace of the real vector space $\mathcal{C}(\mathbb{R})$ of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ having $\mathbb{R}$ as their domain.
   (b) Prove that $D : V \to V$ is a linear operator.
   (c) Determine the null space and range of D.
   (d) Find the matrix representation of D in the basis $\{f_1, f_2\}$.
   (e) Use your response to (d) to determine the matrix representation of D^2 in the basis $\{f_1, f_2\}$.
   (f) Use your response to (d) to show that D is invertible and find $D^{-1} : V \to V$.
   (g) Let $g_1 := f_1 + f_2$ and $g_2 := f_1 - f_2$. Show that $\{g_1, g_2\}$ is a basis of V.
   (h) Find the matrix representation of D in the basis $\{g_1, g_2\}$.
   (i) Use your response to (h) to determine the matrix representation of D^2 in the basis $\{g_1, g_2\}$.

3. Let $p_1, p_2, p_3 : \mathbb{R} \to \mathbb{R}$ be the polynomials defined by $\forall x \in \mathbb{R}$, $p_1(x) := 1$ $p_2(x) := x$, and $p_3(x) := x^2$, V be the vector space of real polynomials of degree at most two, i.e.,

$$V := \{a_1 p_1 + a_2 p_2 + a_3 p_3 \mid a_1, a_2, a_3 \in \mathbb{R}\},$$

   and $L : V \to V$ be defined by

$$\forall p \in V, \forall x \in \mathbb{R} \quad (Lp)(x) := x \frac{d}{dx} p(x) + p(x).$$

   (a) Prove that $L : V \to V$ is a linear operator.

(b) Determine the null space and range of L.

(c) Show that $\{p_1, p_2, p_3\}$ is a basis of V.

(d) Find the matrix representation of L in the basis $\{p_1, p_2, p_3\}$.

(e) Use your response to (d) to determine the matrix representation of L^3 in the basis $\{p_1, p_2, p_3\}$.

(f) Use your response to (e) to compute $L^3(p_2 + p_3)$.

4. Let X be a complex inner product space and $A := \{x_1, x_2, \cdots, x_k\}$ be an orthonormal set of vectors in X. Prove that A is a linearly independent set.

5. Let V be the complex vector space of functions $f : [-\pi, \pi] \to \mathbb{C}$ of the form

$$\forall x \in [-\pi, \pi], \quad f(x) = \alpha_1 + \alpha_2 e^{ix} + \alpha_3 e^{-ix},$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Let $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{C}$ be defined by

$$\forall f, g \in V, \quad \langle f, g \rangle := \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx.$$

Let for all $m \in \{1, 2, 3\}$, $f_m \in V$ be defined by

$$\forall x \in [-\pi, \pi], \quad f_1(x) := 1, \quad f_2(x) := e^{ix}, \quad f_3(x) := e^{-ix},$$

and $L : V \to V$ be defined by

$$\forall f \in V, \forall x \in [-\pi, \pi], \quad (Lf)(x) := \int_{-\pi}^{\pi} \sin(x - t)\, f(t)\, dt.$$

(a) Prove that $\{f_1, f_2, f_3\}$ is a basis of V.

(b) Prove that $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product space.

(c) Construct an orthonormal basis of V by applying the Gram-Schmidt process to $\{f_1, f_2, f_3\}$.

(d) Find the domain of L and show that L is a linear operator.

(e) Find the matrix representation of L in the orthonormal basis you construct in part (c).

(f) Find the null space of L and determine if it is invertible.

(g) Determine whether L is a self-adjoint operator.

(h) Find the eigenvalues of L and obtain an eigenvector for each eigenvalue.