Solutions

Math 503: Quiz # 3

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

---

1. Find all real values of μ for which the following boundary-value problem has a nonzero solution.

$$y'' + \mu^2 y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$$

(30 points) For $\mu \neq 0$:

$y'' + \mu^2 y = 0 \Rightarrow r^2 + \mu^2 = 0 \Rightarrow r = \pm i\mu \Rightarrow y = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$y(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y(x) = c_2 \sin(\mu x)$

$\Rightarrow y'(x) = \mu c_2 \cos(\mu x)$ $\quad y'(\pi) = 0 \Rightarrow \cos(\mu\pi) = 0$

$\Rightarrow \mu\pi = \frac{\pi}{2} + n\pi, \; n \in \mathbb{Z} \Rightarrow \boxed{\mu = n + \frac{1}{2} \text{ with } n \in \mathbb{Z}}$

For $\mu = 0 \Rightarrow y'' = 0 \Rightarrow y = ax + b$

$y(0) = 0 \Rightarrow b = 0 \Rightarrow y(x) = ax$

$\Rightarrow y'(x) = a$ $\quad y'(\pi) = 0 \Rightarrow a = 0 \Rightarrow y(x) = 0$

so for $\mu = 0$ there is no nonzero solution.

30

2. Consider the surfaces S_1 and S_2 defined by

$$S_1 := \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^4 - xyz = 1\}, \qquad S_2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^4 - y^4 + xyz = 1\}$$

2.a) Find a unit normal vector to S_1 at the point $(1, 1, 1)$. (10 points)

Let $\vec{v}_1 := \vec{\nabla}(x^4+y^4-xyz-1) = (4x^3-yz, 4y^3-xz, -xy)$

$\vec{v}_1(1,1,1) = (3, 3, -1)$

$\hat{n}_1 := \dfrac{\vec{v}_1}{|\vec{v}_1|} = \dfrac{1}{\sqrt{9+9+1}}\vec{v}_1 = \left(\dfrac{3}{\sqrt{19}}, \dfrac{3}{\sqrt{19}}, \dfrac{-1}{\sqrt{19}}\right)$ ✓

2.b) Find a unit vector that is tangent to both S_1 and S_2 at the point $(1, 1, 1)$. (20 points)

This common vector $\hat{t}$ must be perpendicular to $\vec{v}_1$ and any normal vector $\vec{v}_2$ to S_2 at $(1,1,1)$

$\vec{v}_2 := \vec{\nabla}(x^4-y^4+xyz-1) = (4x^3+yz, -4y^3+xz, xy)$

$\vec{v}_2(1,1,1) = (5, -3, 1)$

$\hat{t} \perp \vec{v}_1$ and $\hat{t} \perp \vec{v}_2$ so $\tilde{t}$ is along $\vec{v}_1 \times \vec{v}_2$

we can construct such a vector as $\hat{t} = \dfrac{\vec{v}_1 \times \vec{v}_2}{|\vec{v}_1 \times \vec{v}_2|}$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & -1 \\ 5 & -3 & 1 \end{vmatrix} = (0, -8, -24) = -8(0, 1, 3)$$

$$\Rightarrow \hat{t} = \frac{-8(0,1,3)}{|8|\sqrt{1+9}} = \frac{-1}{\sqrt{10}}(0,1,3) = \left(0, \frac{-1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)$$

3. Let $f(x,y) := x^2 + 2axy + by^2$ where a and b are real parameters.

3.a) Find the stationary points of f for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$. (20 points)

$$f_x = 2x + 2ay = 0, \quad f_y = 2ax + 2by = 0$$

$$\Downarrow$$

$$\begin{cases} x + ay = 0 \\ ax + by = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & a \\ a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\det \begin{bmatrix} 1 & a \\ a & b \end{bmatrix} = b - a^2 \neq 0 \Rightarrow x = y = 0$ is the only stationary point.

If $b - a^2 = 0$ i.e., $b = a^2$, $\forall y \in \mathbb{R}$, $(-ay, y)$ is a stationary point.

3.a) Find the necessary and sufficient conditions on a and b such that f has a minimum point. (20 points)

$$f_{xx} = 2, \quad f_{xy} = 2a, \quad f_{yy} = 2b$$

$$\Rightarrow \mathbb{H} = \begin{bmatrix} 2 & 2a \\ 2a & 2b \end{bmatrix}$$

To have a min. point eigenvalues of $\mathbb{H}$ must be both positive $\Rightarrow \begin{cases} \text{trace } \mathbb{H} = 2 + 2b > 0 \\ \det \mathbb{H} = 4(b - a^2) > 0 \end{cases}$

$$\Rightarrow \boxed{b > -1 \text{ and } b > a^2}$$