Solution

Math 503: Quiz # 2

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One hour.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Let V be the complex vector space of functions $f:[-\pi,\pi]\to\mathbb{C}$ of the form: $\forall x\in[-\pi,\pi]$, $f(x)=\alpha_1+\alpha_2 e^{ix}$, where $\alpha_1,\alpha_2\in\mathbb{C}$, and $f_1,f_2\in V$, $\langle\cdot,\cdot\rangle:V^2\to\mathbb{C}$, and $L:V\to V$ be defined by, $\forall x\in[-\pi,\pi]$ and $\forall f,g\in V$,

$$f_1(x):=1,\quad f_2(x):=e^{ix},\quad \langle f,g\rangle:=\int_{-\pi}^{\pi}\sin^2(x)\,\overline{f(x)}\,g(x)\,dx,\quad (Lf)(x):=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f(t+x)\,dt.$$

1.a) Find the matrix representation of L in the basis $\{f_1,f_2\}$. (**20** points)

$\forall f\in V,\ \exists\alpha_1,\alpha_2\in\mathbb{C},\quad f=\alpha_1 f_1+\alpha_2 f_2$

$$\Rightarrow\ (Lf)(x)=\int_{-\frac{\pi}{2}}^{\pi/2}(\alpha_1+\alpha_2 e^{i(t+x)})\,dt=\left(\alpha_1 t+\alpha_2 e^{ix}\frac{e^{it}}{i}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\alpha_1\pi+\alpha_2 e^{ix}\underbrace{\left(\frac{e^{\frac{i\pi}{2}}-e^{-\frac{i\pi}{2}}}{i}\right)}_{2\sin(\frac{\pi}{2})=2}=\alpha_1\pi+2\alpha_2 e^{ix}$$

$$=\alpha_1\pi f_1(x)+2\alpha_2 f_2(x)$$

$$\Rightarrow\ \boxed{L(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\pi f_1+2\alpha_2 f_2}\ \Rightarrow\ \begin{matrix}Lf_1=\pi f_1+0f_2\\ Lf_2=0f_1+2f_2\end{matrix}$$

$$Lf_i=\sum_{j=1}^{2}L_{ji}f_j\ \Rightarrow\quad \mathbb{L}=(L_{ji})=\begin{pmatrix}\pi&0\\0&2\end{pmatrix}$$

1.b) Construct an orthonormal basis of V by applying the Gram-Schmidt process to $\{f_1, f_2\}$. (25 points)

$$e_1 := \frac{f_1}{\|f_1\|}, \quad \|f_1\|^2 = \langle f_1, f_2\rangle = \int_{-\pi}^{\pi} \sin^2(x)\, \underbrace{|f_1(x)|^2}_{1} dx$$

$$\Rightarrow \|f_1\|^2 = \int_{-\pi}^{\pi} \left(\frac{1-\cos(2x)}{2}\right) dx = \pi - \frac{1}{2} \underbrace{\frac{\sin(2x)}{2}\Big|_{-\pi}^{\pi}}_{0} = \pi$$

$$\Rightarrow \|f_1\| = \sqrt{\pi} \Rightarrow \boxed{e_1 := \frac{f_1}{\sqrt{\pi}}}$$

$$v_2 := f_2 - \langle e_1, f_2\rangle e_1$$

$$\langle e_1, f_2\rangle = \int_{-\pi}^{\pi} \sin^2 x \left(\frac{\overline{f_1(x)}}{\sqrt{\pi}}\right) f_2(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} e^{ix} \sin^2 x\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} e^{ix} \left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{\sqrt{\pi}} \left(\underbrace{\int_{-\pi}^{\pi} \frac{e^{ix}}{2} dx}_{I_1} - \frac{1}{2} \underbrace{\int_{-\pi} e^{ix} \cos(2x)\, d}_{I_2} \right.$$

$$I_1 = \frac{e^{ix}}{2i}\Big|_{-\pi}^{\pi} = \frac{e^{i\pi} - e^{-i\pi}}{2i} = \sin(\pi) = 0$$

$$I_2 = \int_{-\pi}^{\pi} e^{ix} \left(\frac{e^{2ix} + e^{-2ix}}{2}\right) dx = \frac{1}{2} \int_{-\pi}^{\pi} (e^{3ix} + e^{-ix})\, dx = \frac{1}{2}\left(\frac{e^{3ix}}{3i} + \frac{e^{-ix}}{-i}\right|$$

$$= \frac{1}{3}\left(\frac{e^{3i\pi} - e^{-3i\pi}}{2i} + \frac{e^{-i\pi} - e^{i\pi}}{2i}\right) = \frac{1}{3}\left(\overset{\to 0}{\sin(3\pi)} - \overset{\to 0}{\sin(\pi)}\right) = 0$$

so $\langle e_1, f_2\rangle = 0 \Rightarrow e_2 = \frac{v_2}{\|v_2\|} = \frac{f_2}{\|f_2\|}$

$$\|f_2\|^2 = \int_{-\pi}^{\pi} \sin^2 x\, \underbrace{e^{-ix} e^{ix}}_{1} dx = \int_{-\pi}^{\pi} \sin^2 x\, dx = \|f_1\|^2 = \pi \Rightarrow \boxed{e_2 = \frac{f_2}{\sqrt{\pi}}}$$

$\{e_1, e_2\}$ is an orthonormal basis.

1.c) Determine whether L is a self-adjoint operator and justify your response. (15 points)

$\forall f, g \in V, \quad \exists \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C},$

$f = \alpha_1 f_1 + \alpha_2 f_2, \quad g = \beta_1 f_1 + \beta_2 f_2$

$\langle f, Lg \rangle = \langle \alpha_1 f_1 + \alpha_2 f_2, L(\beta_1 f_1 + \beta_2 f_2) \rangle$

$= \langle \alpha_1 f_1 + \alpha_2 f_2, \pi \beta_1 f_1 + 2\beta_2 f_2 \rangle$

$= \bar{\alpha}_1 \pi \beta_1 \langle f_1, f_1 \rangle + \bar{\alpha}_1 2\beta_2 \langle f_1, f_2 \rangle +$

$\bar{\alpha}_2 \pi \beta_1 \langle f_2, f_1 \rangle + \bar{\alpha}_2 2\beta_2 \langle f_2, f_2 \rangle$

As we showed in 1.b $\langle f_1, f_2 \rangle = 0 \Rightarrow \langle f_2, f_1 \rangle = 0$

$\langle f_1, f_1 \rangle = \langle f_2, f_2 \rangle = \pi$,

$\Rightarrow \langle f, Lg \rangle = \pi(\pi \bar{\alpha}_1 \beta_1 + 2\bar{\alpha}_2 \beta_2)$

Similarly we have $\langle Lf, g \rangle = \overline{\langle g, Lf \rangle} = \overline{\pi(\pi \bar{\beta}_1 \alpha_1 + 2\bar{\beta}_2 \alpha_2)}$

$= \pi(\pi \beta_1 \bar{\alpha}_1 + 2\beta_2 \bar{\alpha}_2)$

$= \langle f, Lg \rangle$

So L is self-adjoint.

1.d) Find the eigenvalues of L. (10 points)

$Lf = \lambda f, \quad \forall f \in V, \exists \alpha_1, \alpha_2 \in \mathbb{C}, \; f = \alpha_1 f_1 + \alpha_2 f_2$

$\Rightarrow L(\alpha_1 f_1 + \alpha_2 f_2) = \lambda(\alpha_1 f_1 + \alpha_2 f_2) \Rightarrow$

$\pi \alpha_1 f_1 + 2\alpha_2 f_2 = \lambda \alpha_1 f_1 + \lambda \alpha_2 f_2$

$\Rightarrow (\pi - \lambda)\alpha_1 f_1 + (2 - \lambda)\alpha_2 f_2 = 0$

$f \neq 0 \Rightarrow$ either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. f_1 & f_2 are linearly-indep $\Rightarrow \pi - \lambda = 0$ & $2 - \lambda = 0$

$\Rightarrow \lambda = \pi$ or 2. The eigenvalues are 2 and π.

2. Find the general solution of $y'' + y = \frac{1}{\sin x}$. (30 points)

$$y'' + y = 0 \implies r^2 + 1 = 0 \implies r = \pm i$$

$$\implies y_1(x) = e^{ix}, \quad y_2(x) = e^{-ix}, \quad \forall x \in \mathbb{R}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_a^x G(x,y) \left(\frac{1}{\sin y} \right) dy$$

$$G(x,y) = \frac{\begin{vmatrix} e^{iy} & e^{-iy} \\ e^{ix} & e^{-ix} \end{vmatrix}}{\begin{vmatrix} e^{iy} & e^{-iy} \\ ie^{iy} & -ie^{-iy} \end{vmatrix}} = \frac{e^{i(y-x)} - e^{-i(y-x)}}{-2i} = \sin(x-y)$$

$$\int_a^x G(x,y) \frac{1}{\sin y} dy = \int_a^x \frac{\sin(x-y)}{\sin y} dy = \int_a^x \frac{\sin x \,\mathrm{Cn}\, y - \sin y \,\mathrm{Cn}\, x}{\sin y} dy$$

$$= \sin x \int_a^x \frac{\mathrm{Cn}\, y \, dy}{\sin y} - \mathrm{Cn}\, x \int_a^x dy$$

$$u = \sin y \qquad du = \mathrm{Cn}\, y \, dy$$

$$\int \frac{\mathrm{Cn}\, y \, dy}{\sin y} = \int \frac{du}{u} = \ln|u| + c = \ln|\sin(y)| + c$$

$$\implies \int_a^x G(x,y) \frac{1}{\sin y} dy = \sin x \left[\ln|\sin(x)| - \ln|\sin(a)| \right] - \mathrm{Cn}\, x (x - a)$$

choose $a = \pi/2 \implies \sin(a) = 1 \implies \ln|\sin(a)| = 0$

$\Downarrow$

$$y(x) = c_1 e^{ix} + c_2 e^{-ix} + \sin x \ln|\sin(x)| - \mathrm{Cn}\, x (x - \pi/2)$$

$$= a_1 \sin x + a_2 \mathrm{Cn}\, x + \sin x \ln|\sin(x)| - x \,\mathrm{Cn}\, x$$

$a_1, a_2 \in \mathbb{R}$.