Solutions

Math 503: Quiz # 1

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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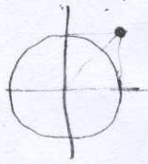
1. Find all possible values of the real part of $(i+1)^{i-1}$. (20 points)



$$(i+1)^{i-1} = e^{(i-1)\ln(i+1)}$$

$$i+1 = \sqrt{2}\, e^{\frac{i\pi}{4} + 2\pi n i} \quad n \in \mathbb{Z} \Rightarrow \ln(i+1) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right), \quad n \in \mathbb{Z}$$

$$\Rightarrow (i-1)\ln(i+1) = (i-1)\ln\sqrt{2} + (i-1)\, i\left(\frac{\pi}{4} + 2\pi n\right)$$

$$= -\ln\sqrt{2} - \left(\frac{\pi}{4} + 2\pi n\right) + i\left(\ln\sqrt{2} - \frac{\pi}{4} - 2\pi n\right)$$

$$\Rightarrow (i+1)^{i-1} = e^{-\left[\ln\sqrt{2} + \left(\frac{\pi}{4} + 2\pi n\right)\right] + i\left[\ln\sqrt{2} - \frac{\pi}{4} - 2\pi n\right]}$$

$$\Rightarrow \boxed{\operatorname{Re}\left[(i+1)^{i-1}\right] = e^{-\left(\ln\sqrt{2} + \frac{\pi}{4} + 2\pi n\right)} \cos\left(\ln\sqrt{2} - \frac{\pi}{4} - 2\pi n\right) \quad \text{when } n \in \mathbb{Z}}$$

2.a Let $a, b, c \in \mathbb{C}$ be such that $a \neq 0$ and $b \neq 0$. Prove that $(a^b)^c = a^{bc}$. (10 points)

$$(a^b)^c = (e^{b \ln a})^c = e^{c \ln (e^{b \ln a})}$$

$$= e^{c(b \ln a)} = e^{cb \ln a} = a^{cb} = a^{b c}.$$

2.b Show that for all $z, w \in \mathbb{C}$, $\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w$. (10 points)

$$\sinh z \cosh w + \cosh z \sinh w = \frac{(e^z - e^{-z})(e^w + e^{-w})}{4} + \frac{(e^z + e^{-z})(e^w - e^{-w})}{4}$$

$$= \frac{1}{4}\Big[e^{z+w} - e^{-z+w} + e^{z-w} - e^{-z-w} + e^{z+w} - e^{z-w} + e^{-z+w} - e^{-z-w} \Big]$$

$$= \frac{1}{4}\left(2e^{z+w} - 2e^{-(z+w)} \right)$$

$$= \sinh(z+w)$$

3. Let V be a complex vector space, and U and W are subspaces of V. Show that the intersection of U and W, i.e., $U \cap W := \{a \in V \mid a \in U \text{ and } a \in W\}$, is also a subspace of V. (15 points)

Clearly $U \cap W \subseteq U \subseteq V \Rightarrow U \cap W \subseteq V$.

$\forall \alpha, \beta \in \mathbb{C}$, $\forall v_1, v_2 \in U \cap W \Rightarrow v_1 \in U, v_2 \in U \Rightarrow$

(1) $\boxed{\alpha v_1 + \beta v_2 \in U}$ because U is a subspace

Also $v_1 \in W$, $v_2 \in W \Rightarrow \boxed{\alpha v_1 + \beta v_2 \in W}$ because W is a subspace
(2)

(1) & (2) $\Rightarrow \alpha v_1 + \beta v_2 \in U \cap W \Rightarrow U \cap W$ is a subspace.

4. Consider the complex vector space of all complex 2×2 matrices:

$$M(2, \mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{C} \right\},$$

and let $L : M(2, \mathbb{C}) \to \mathbb{C}$ be defined by $\forall \mathbb{A} \in M(2, \mathbb{C})$, $L\mathbb{A} := \det(\mathbb{A})$, where

$$\forall a, b, c, d \in \mathbb{C}, \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc.$$

Is L a linear operator? Why? (15 points)

No: because $2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\Rightarrow L\left(2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = L\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$

but $2L\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$

So $L\left(2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 2L\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So L is not linear.

5. Consider the complex vector space $C(\mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is a function }\}$ and let $f, g, h \in C(\mathbb{C})$ be defined by

$$\forall x \in \mathbb{R}, \quad f(x) := e^{ix}, \quad g(x) := e^{-ix}, \quad h(x) = e^{2ix}$$

Prove that $\{f, g, h\}$ is a linearly independent subset of $C(\mathbb{C})$. (30 points)

$\forall \alpha, \beta, \gamma \in \mathbb{C}, \quad \alpha f + \beta g + \gamma h = 0$

$\Rightarrow \forall x \in \mathbb{R}, \quad \alpha f(x) + \beta g(x) + \gamma h(x) = 0$

$\Rightarrow \forall x \in \mathbb{R}, \quad \alpha e^{ix} + \beta e^{-ix} + \gamma e^{2ix} = 0$

$\Downarrow$

$e^{-ix}\left[\alpha e^{2ix} + \beta + \gamma e^{3ix}\right] = 0$

$\forall x \in \mathbb{R}, \ e^{-ix} \neq 0 \qquad \Downarrow$

$(*) \quad \boxed{\beta + \alpha e^{2ix} + \gamma e^{3ix} = 0, \ \forall x \in \mathbb{R}}$

For $x = 0$

$(*) \Rightarrow \quad \beta + \alpha + \gamma = 0 \quad (1) \quad \Rightarrow \quad 2\beta = (i-1)\gamma$

for $x = \frac{\pi}{2}$

$(*) \Rightarrow \quad \beta - \alpha - i\gamma = 0 \quad (2) \quad \Rightarrow \quad 2i\gamma = 0 \quad \Rightarrow \boxed{\gamma = 0}$

For $x = -\frac{\pi}{2}$

$\Rightarrow \quad \beta - \alpha + i\gamma = 0 \quad (3) \qquad \boxed{\beta = 0}$

$(1) \Rightarrow \boxed{\alpha = 0}$

So $\alpha = \beta = \gamma = 0 \Rightarrow \{f, g, h\}$ is linearly indep.