Math 503: Final Exam

Fall 2007

You are two and half hours.

Problem 1. Let $\mathcal{M}$ denote the complex vector space of 3×2 complex matrices, V be the subset of $\mathcal{M}$ consisting of the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \\ c & 0 \end{pmatrix}$ where $a, b, c \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{C}$ be defined by: $\langle A, B \rangle := \mathrm{tr}(A^\dagger B)$, where $A, B \in V$, "tr" stands for the trace of a square matrix (sum of its diagonal entries) and $A^\dagger = \bar{A}^t$ is the transpose of the complex-conjugate of A.

1.a) Show that V is a subspace of $\mathcal{M}$. (3 points)

1.b) Find a linear operator $L : \mathcal{M} \to \mathcal{M}$ such that V is the null space of L. (3 points)

1.c) Use the definition of an inner product on a complex vector space to show that $\langle \cdot, \cdot \rangle$ is an inner product on V. (9 points)

1.d) Let $A_1 := \begin{pmatrix} i & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 := \begin{pmatrix} 1 & 0 \\ 0 & i \\ 1 & 0 \end{pmatrix}$, $A_3 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ i & 0 \end{pmatrix}$. Show that $\{A_1, A_2, A_3\}$ is a basis of V. (7 points)

1.e) Perform the Gram-Schmidt process on $\{A_1, A_2, A_3\}$ to construct an orthonormal basis for the inner product space $(V, \langle \cdot, \cdot \rangle)$. (8 points)

Problem 2. Verify the Stokes' theorem for the surface

$$S := \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{1 - x^2 - y^2} \right\}$$

and the vector field $\mathbf{F}(x, y, z) := -y\mathbf{i} + x\mathbf{j} + xy\mathbf{k}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the x-, y-, and z-axes. (15 points)

Problem 3. Find a power series solution for the following integral equation about $x = 0$.

$$\int_0^x e^{\frac{t}{x}} \, y(t) \, dt + y(x) = 1.$$

You may express your solution in terms of $c_n := \int_0^1 t^n e^t dt$ where $n \in \mathbb{N}$. Note that $\int_0^x e^{\frac{t}{x}} \, t^n dt = c_n x^{n+1}$, $c_0 = e - 1$, and $c_n = e - n c_{n-1}$ for all $n \geq 1$. (15 points)

Problem 4. Find the general form of a stationary point of the following functional with fixed boundary conditions. (10 points)

$$\mathcal{F}[y(x)] := \int_0^1 \left[e^{y'(x)} + y(x) \right] dx.$$

Problem 5. Find $u(x,y,t)$ for all $0 \le x \le \pi, \quad 0 \le y \le \pi, \quad t \ge 0$ such that

$$\begin{aligned}
&u_t = u_{xx} + u_{yy}, && \text{for } \ 0 \le x \le \pi, \ \ 0 \le y \le \pi, \ t \ge 0,\\
&u(0,y,t) = u(\pi,y,t) = 0, && \text{for } \ 0 \le y \le \pi, \ \ t \ge 0,\\
&u_y(x,0,t) = u_y(x,\pi,t) = 0, && \text{for } \ 0 \le x \le \pi, \ \ t \ge 0,\\
&u(x,y,0) = 1, && \text{for } \ 0 \le x \le \pi, \ \ 0 \le y \le \pi.
\end{aligned}$$

(15 points)

Problem 6. Let $u(x,y,t)$ be the solution of the following initial-value problem.

$$\begin{aligned}
&u_t = u_{xy}, \quad \text{for } \ x, y \in \mathbb{R}, \ t \ge 0,\\
&u(x,y,0) = \begin{cases} 1 & \text{for } \ |x| \le 1, \ |y| \le 1 \\ 0 & \text{for } \ |x| > 1, \ |y| > 1. \end{cases}
\end{aligned}$$

Use the method of Fourier transform to express $u(x,y,t)$ in the form

$$u(x,y,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y,t,k_1,k_2) \, dk_1 \, dk_2,$$

and obtain an explicit expression for $f(x,y,t,k_1,k_2)$. (15 points)