Math 503, Fall 2006
Assignment for October 12-15

- Read pages 333-348 of the textbook (Kreyszig, 9th Edition)
- Solve Problems 6, 10, 14, 29, 30 on pages 338-339 of the textbook and the following problems.

1. Let X be a complex inner product space and $A := \{x_1, x_2, \cdots, x_k\}$ be an orthonormal set of vectors in X. Prove that A is a linearly independent set.
2. Let V be the complex vector space of functions $f : [-\pi, \pi] \to \mathbb{C}$ of the form

$$\forall x \in [-\pi, \pi], \quad f(x) = \alpha_1 + \alpha_2 e^{ix} + \alpha_3 e^{-ix},$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Let $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{C}$ be defined by

$$\forall f, g \in V, \quad \langle f, g \rangle := \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx.$$

Let for all $m \in \{1, 2, 3\}$, $f_m \in V$ be defined by

$$\forall x \in [-\pi, \pi], \quad f_1(x) := 1, \quad f_2(x) := e^{ix}, \quad f_3(x) := e^{-ix},$$

and $L : V \to V$ be defined by

$$\forall f \in V, \forall x \in [-\pi, \pi], \quad (Lf)(x) := \int_{-\pi}^{\pi} \sin(x - t)\, f(t)\, dt.$$

(a) Prove that $\{f_1, f_2, f_3\}$ is a basis of V.

(b) Prove that $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product space.

(c) Construct an orthonormal basis of V by applying the Gram-Schmidt process to $\{f_1, f_2, f_3\}$.

(d) Find the domain of L and show that L is a linear operator.

(e) Find the matrix representation of L in the orthonormal basis you construct in part (c).

(f) Find the null space of L and determine if it is invertible.

(g) Determine whether L is a self-adjoint operator.

(h) Find the eigenvalues of L and obtain an eigenvector for each eigenvalue.