Math 503, Fall 2006
Assignment for October 05-09

Solve Problems 6, 8, 9 on page 329 of the textbook (Kreyszig, 9th Edition) and the following problems.

1. Let V and W be complex vector spaces and $L : V \to W$ be a linear operator.
   (a) Prove that the null space of L, i.e., $\text{null}(L) := \{v \in V \mid Lv = 0\}$, is a subspace of V.
   (b) Prove that the range of L, i.e., $\text{range}(L) := \{w \in W \mid \exists v \in V, w = Lv\}$, is a subspace of W.

2. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined by $\forall x \in \mathbb{R}$, $f_1(x) := \sin x$ and $f_2(x) := \cos x$, $V := \{a_1 f_1 + a_2 f_2 | a_1, a_2 \in \mathbb{R}\}$, and $D : V \to V$ denote the differentiation.
   (a) Prove that V is a subspace of the real vector space $\mathcal{C}(\mathbb{R})$ of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ having $\mathbb{R}$ as their domain.
   (b) Prove that $D : V \to V$ is a linear operator.
   (c) Determine the null space and range of D.
   (d) Find the matrix representation of D in the basis $\{f_1, f_2\}$.
   (e) Use your response to (d) to determine the matrix representation of D^2 in the basis $\{f_1, f_2\}$.
   (f) Use your response to (d) to show that D is invertible and find $D^{-1} : V \to V$.
   (g) Let $g_1 := f_1 + f_2$ and $g_2 := f_1 - f_2$. Show that $\{g_1, g_2\}$ is a basis of V.
   (h) Find the matrix representation of D in the basis $\{g_1, g_2\}$.
   (i) Use your response to (h) to determine the matrix representation of D^2 in the basis $\{g_1, g_2\}$.

3. Let $p_1, p_2, p_3 : \mathbb{R} \to \mathbb{R}$ be the polynomials defined by $\forall x \in \mathbb{R}$, $p_1(x) := 1$ $p_2(x) := x$, and $p_3(x) := x^2$, V be the vector space of real polynomials of degree at most two, i.e.,

$$V := \{a_1 p_1 + a_2 p_2 + a_3 p_3 \mid a_1, a_2, a_3 \in \mathbb{R} \},$$

   and $L : V \to V$ be defined by

$$\forall p \in V, \forall x \in \mathbb{R} \quad (Lp)(x) := x \frac{d}{dx} p(x) + p(x).$$

   (a) Prove that $L : V \to V$ is a linear operator.
   (b) Determine the null space and range of L.
   (c) Show that $\{p_1, p_2, p_3\}$ is a basis of V.
   (d) Find the matrix representation of L in the basis $\{p_1, p_2, p_3\}$.
   (e) Use your response to (d) to determine the matrix representation of L^3 in the basis $\{p_1, p_2, p_3\}$.
   (f) Use your response to (e) to compute $L^3(p_2 + p_3)$.