Solutions.

Math 503: Quiz # 4

Fall 2006

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	

- You have 60 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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Problem 1. State the Divergence theorem in $\mathbf{R}^3$ and explain how it is used to derive Green's first identity, i.e.,

$$\iiint_V (f\nabla^2 g + \vec{\nabla} f\cdot\vec{\nabla} g)\, dV = \iint_S f\,\frac{\partial g}{\partial n}\, dS.$$

(20 points)

Let V be a closed bounded region in $\mathbb{R}^3$ and S be the boundary surface of V. Let $\vec{A}:\mathbb{R}^3\to\mathbb{R}^3$ be a vector field that is piecewise smooth in a region $U\subseteq\mathbb{R}^3$ such that $V\cup S\subset U$. Then

$$\iiint_V \vec{\nabla}\cdot\vec{A}\, dV = \iint_S \vec{A}\cdot\hat{n}\, dS \qquad (\mathrm{I})$$

where $\hat{n}$ is unit outward normal vector to S.

Let $f:\mathbb{R}^3\to\mathbb{R}$ and $g:\mathbb{R}^3\to\mathbb{R}$ be respectively differentiable and twice differentiable functions in U. Setting $\vec{A}=f\vec{\nabla}g$ in (I) and using $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}(f\vec{\nabla}g) = f\vec{\nabla}^2 g + \vec{\nabla}f\cdot\vec{\nabla}g$ we have

$$\iiint_V (f\nabla^2 g + \vec{\nabla}f\cdot\vec{\nabla}g)\, dV = \iint_S f\,\vec{\nabla}g\cdot\hat{n}\, dS \quad \text{but } \vec{\nabla}g\cdot\hat{n} = \frac{\partial g}{\partial n}$$

(Directional derivative of g along $\hat{n}$)

$$= \iint_S f\,\frac{\partial g}{\partial n}\, dS.$$

Problem 2. Use Stokes' theorem to evaluate the absolute value of the line integral $\oint_C \vec{F}\cdot d\vec{x}$ where $\vec{F} := y\hat{\imath} + xy^3\hat{\jmath} - zy^3\hat{k}$, and C is the circle defined by $x^2+y^2=1$ and $z=1$. (30 points)

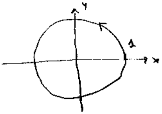
$$\left|\oint_C \vec{F}\cdot d\vec{x}\right| = \left|\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds\right|$$

$C = \partial S$

choose $\quad S := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1,\ z=1\}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & xy^3 & -zy^3 \end{vmatrix} = -3y^2z\,\hat{\imath} - (0)\cdot\hat{\jmath} + (y^3-1)\hat{k}$$

$$= -3y^2z\,\hat{\imath} + (y^3-1)\hat{k}$$



$\hat{n} = \hat{k}$

$(\vec{\nabla}\times\vec{F})\cdot\hat{k} = y^3-1$

In cylindrical coordinat $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$ds = dx\,dy = r\,dr\,d\theta$

$$\Rightarrow \left|\oint_C \vec{F}\cdot d\vec{x}\right| = \left|\int_0^{2\pi}\int_0^1 (r^3\sin^3\theta - 1)\, r\,dr\,d\theta\right|$$

$$= \left|\int_0^{2\pi}\left[\frac{r^5}{5}\sin^3\theta - \frac{r^2}{2}\Big|_0^1\right] d\theta\right|$$

$$= \left|\int_0^{2\pi}\left(\frac{\sin^3\theta}{5} - \frac{1}{2}\right) d\theta\right| \qquad \begin{aligned}\sin^3\theta &= \sin^2\theta\cdot\sin\theta \\ &= (1-\cos^2\theta)\sin\theta\end{aligned}$$

$$= \left|\frac{1}{5}\int_0^{2\pi}(\sin\theta - \sin\theta\cos^2\theta)\,d\theta - \frac{\theta}{2}\Big|_0^{2\pi}\right| \qquad \begin{aligned}\cos\theta &= u \\ -\sin\theta\, d\theta &= du\end{aligned}$$

$$= \left|\frac{1}{5}\Big(\underbrace{-\cos\theta\Big|_0^{2\pi}}_{0} + \underbrace{\int_1^1 u^2\,du}_{0}\Big) - \pi\right|$$

$$= \pi$$

Problem 3. Find geodesics in the Euclidean space $\mathbb{R}^3$, i.e., determine the functions $y = y(x)$ and $z = z(x)$ such that the curve defined by:

$$\vec{X}(x) := \begin{pmatrix} x \\ y(x) \\ z(x) \end{pmatrix}$$

that joins two fixed points in $\mathbb{R}^3$ has the shortest length. (30 points)

$$dl^2 = dx^2 + dy^2 + dz^2 = (1 + y'^2 + z'^2)\, dx^2$$

$$l[\vec{X}] = \int_{x_1}^{x_2} \underbrace{\sqrt{1+y'^2+z'^2}}_{F}\, dx^2$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0, \qquad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2+z'^2}}, \qquad \frac{\partial F}{\partial z'} = \frac{z'}{\sqrt{1+y'^2+z'^2}}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2+z'^2}} = c_1 \qquad \Rightarrow \frac{y'^2}{1+y'^2+z'^2} = c_1^2$$

c_1 const.

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0 \Rightarrow \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) = 0$$

$$\Rightarrow \frac{z'}{\sqrt{1+y'^2+z'^2}} = c_2 \qquad \Rightarrow \frac{z'^2}{1+y'^2+z'^2} = c_2^2$$

c_2 const.

$$\Rightarrow \frac{y'}{c_1} = \frac{z'}{c_2} \Rightarrow z' = \frac{c_2}{c_1} y'$$

$$\Rightarrow \frac{y'^2}{1 + (1 + \frac{c_2^2}{c_1^2})y_1^2} = c_1^2 \Rightarrow y'^2 = \frac{c_1^2}{1 - (1+\frac{c_2^2}{c_1^2})c_1^2} \Rightarrow y' = \underset{\text{const}}{m_1}$$

$$\Rightarrow z' = \frac{c_2}{c_1} m_1 =: \underset{\text{const}}{m_2} \qquad \text{So} \quad \begin{cases} y = m_1 x + b_1 \\ z = m_2 x + b_2 \end{cases}$$

$$\Rightarrow \vec{X}(x) = \begin{pmatrix} x \\ m_1 x + b_1 \\ m_2 x + b_2 \end{pmatrix}.$$

This is a line in $\mathbb{R}^3$. m_1, m_2, b_1, b_2 are fixed by requiring this line to joint

Problem 4. A pendulum is constructed by attaching a particle of mass m to a (massless) spring of spring constant k. The other end of the spring is attached to the origin of the x-y plane and the particle is allowed to move only in this plane. Using a set of appropriate polar coordinates, (r, θ), we can express the kinetic and potential energies of the system in the form:

$$KE = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right), \qquad PE = -mgr\sin\theta + \frac{k}{2}(r-d)^2,$$

where a dot means a time-derivative, and g and d are positive real constants. Find the equations of motion of the this system using Hamilton's Least Action Principle. Express these equations as

$$\ddot{r} = f_1(r, \dot{r}, \theta, \dot{\theta}), \qquad \ddot{\theta} = f_2(r, \dot{r}, \theta, \dot{\theta}),$$

and find the form of the functions f_1 and f_2. (20 points)

$$S = \int_{t_1}^{t_2} (KE - PE)\,dt = \int_{t_1}^{t_2} dt \underbrace{\left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\sin\theta - \frac{k}{2}(r-d)^2\right]}_{L}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\sin\theta - k(r-d)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \theta} = mgr\cos\theta \ , \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \Rightarrow mr\dot{\theta}^2 + mg\sin\theta - k(r-d) - m\ddot{r} = 0$$

$$\Rightarrow \boxed{\ddot{r} = \underbrace{r\dot{\theta}^2 + g\sin\theta - \frac{k}{m}(r-d)}_{f_1}}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow mgr\cos\theta - m\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

$$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = gr\cos\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = \underbrace{\frac{1}{r}\left[g\cos\theta - 2\dot{r}\dot{\theta}\right]}_{f_2}}$$