Math 450/586: Quiz # 2

Fall 2009

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 75 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask, 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Give the definition of the following objects. (25 points)

1.a) A linear map:

1.b) A bilinear map:

1.c) Algebraic dual of a vector space:

1.d) Dual basis:

1.e) Tensor product of two vector spaces V and W:

2. Let V be a real vector space.

a) Give the definition of the natural injection $\mathcal{I} : V \to V^{**}$ of V into V^{**}. (10 points)

b) Prove that $\forall v_1, v_2 \in V$ and $\forall \alpha^1, \alpha^2 \in \mathbb{R}$, $\mathcal{I}(\alpha^1 v_1 + \alpha^2 v_2) = \alpha^1 \mathcal{I}(v_1) + \alpha^2 \mathcal{I}(v_2)$. (10 points)

c) Prove that $\mathcal{I}$ is one-to-one. (**10 Bonus points**)

3. Let V be a finite-dimensional vector space.

3.a) Let $v \in V$ and $\sigma \in V^*$. Give the definition of $v \otimes \sigma$. (5 points)

Hint: Recall that $\forall u \in V,\ \forall \omega \in V^*,\ v(\omega) := \omega(v)$.

3.b) Show that $v \otimes \sigma \in \mathcal{L}(V^*, V; \mathbb{R})$, i.e., it is a real-valued bilinear map with domain $V^* \times V$. (20 points)

3.c) Let $\mathscr{B} = \{e_1, e_2, \cdots e_n\}$ be a basis of V, and $\mathscr{B}^* = \{\epsilon^1, \epsilon^2, \cdots \epsilon^n\}$ be the basis dual to $\mathscr{B}$. Show that $\mathscr{E} := \{e_i \otimes \epsilon^j | i, j \in I_n\}$ is a basis of $V \otimes V^*$, where $I_n := \{1, 2, \cdots, n\}$. (30 points)