Math 450-586: Midterm Exam 2

Fall 2009

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One and half hours (90 minutes).
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1. Let (M_1, d_1) and (M_2, d_2) be metric spaces, and $f : M_1 \to M_2$ be an isometry.

1.a) Show that f is continuous. (10 points)

1.b) Is f a homeomorphism? Why? (5 points)

Problem 2. Let $\mathcal{M}$ denote the set of 2×2 matrices, $N := \{A \in \mathcal{M} \mid \det(A) > 1\}$, and $f : \mathcal{M} \to \mathbb{R}^4$ be defined by: For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$, $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (a, b, c, d)$.

2.a) Given that f is a bijection, determine a topology on $\mathcal{M}$ such that f is a homeomorphism. You must give the open subsets in this topology and show that f has this property. (10 points)

2.b) Giving $\mathcal{M}$ this topology, we can turn N into a topological space using the induced (subspace) topology. Show that this makes N into a topological manifold. (10 points)

Problem 3. Let $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$, be defined by $\forall x \in \mathbb{R}$, $\phi_1(x) := x$ and $\phi_2(x) := x^5$. Let $M_1 = \mathbb{R}$ be the topological manifold having $\{(\mathbb{R}, \phi_1)\}$ as an atlas, and M_2 be the topological manifolds having $\{(\mathbb{R}, \phi_2)\}$ as an atlas.

3.a) Is M_2 a C^∞-manifold? Why? (5 points)

3.b) Let $f : M_2 \to M_2$ be defined by $\forall x \in \mathbb{R}$, $f(x) := x^{1/5}$. Is f a C^∞-function? (10 points)

3.c) Let $h : M_1 \to M_2$ be defined by $\forall x \in \mathbb{R}$, $h(x) := x^{1/5}$. Is g a C^∞-function? (10 points)

Problem 4. Let M be a three-dimensional C^∞-manifold with a coordinate chart (U_α, ϕ_α), and $F : M \to M$ be a function such that $F(U_\alpha) \subseteq U_\alpha$. Let X and ω be respectively a vector field and a differential form, and $f : M \to \mathbb{R}$ be function. Suppose that X, ω, F, and f have the following local expressions in the coordinate chart (U_α, ϕ_α). For all $p \in U_\alpha$ with $\phi(p) = (x^1, x^2, x^3)$,

$$
\begin{aligned}
X(p) &= (x^2 - x^3)\frac{\partial}{\partial x^1} + (x^3 - x^1)\frac{\partial}{\partial x^2} + (x^1 - x^2)\frac{\partial}{\partial x^3}, \\
\omega(p) &= \sin(x^3)\, dx^1 \wedge dx^2 + \cos(x^1)\, dx^2 \wedge dx^3, \\
F(p) &= \phi_\alpha^{-1}(x^3, 2x^1, 3x^2), \qquad f(p) = (x^1 + x^2 - x^3)^2.
\end{aligned}
$$

Calculate the following quantities.

4.a) $X(f)$. (10 points)

4.b) $(F^*(X))(f)$. (10 points)

4.c) $d\omega$. (10 points)

4.d) $i_X(\omega)$. (10 points)

Hint: First express all the components ω_{ij} of ω.