Math 450-586: Midterm Exam 1

Fall 2009

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One and half hours (90 minutes).
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1. Let V be a real vector space of dimension $n < \infty$, $\{e_i\}$ be a basis of V with dual basis $\{\epsilon^i\}$, and $T = T^{ij}_{pq}\, e_i \otimes e_j \otimes \epsilon^p \otimes \epsilon^q \in V^2_2$, i.e., T is a tensor of type (2,2).

1.a) Derive the transformation rule for the components T^{ij}_{pq} of T under the basis transformations: $e_i \to \tilde{e}_i = a^j_i e_j$, where (a^j_i) is an $n \times n$ invertible matrix. (15 points)

1.b) Show that T^{ij}_{ji} does not change under the above basis transformations, i.e., it is a scalar invariant of T. (10 points)

Problem 2. Let V be a real vector space and $\mathcal{I} : V \to V^{**}$ be the natural injection of V into its second dual. Give the definition of $\mathcal{I}$ and prove that it is one-to-one. (15 points)

Problem 3. Let V be a real vector space of dimension $n < \infty$, V^* be its dual space, and $\mathfrak{S}$ and $\mathfrak{A}$ be the operation of symmetrization and anti-symmetrization of tensors. Suppose that $u, v \in V$ and $\omega, \tau \in V^*$ be such that

$$\omega(u) = 1, \quad \tau(u) = -2, \quad \omega(v) = 2, \quad \tau(v) = -1,$$

and $T := \mathfrak{S}(u \otimes v) \otimes \mathfrak{A}(\omega \otimes \tau)$. Compute $T(\omega + \tau, -\tau, u, u - v)$. (25 points)

Problem 4. Let $\langle\cdot|\cdot\rangle$ be an inner product on a finite-dimensional real vector space V of dimension n, $\{e_i\}$ be a basis of V with dual basis $\{\epsilon^i\}$, and $g : V \times V \to \mathbb{R}$ defined by $\forall u, v \in V$, $g(u, v) := \langle u|v\rangle$.

4.a) Show that g a symmetric tensor of type (0,2). (10 points)

Note: You only need to show that g is bilinear and symmetric.

4.b) Let $g_{ij}, h^{ij} \in \mathbb{R}$ be such that $g = g_{ij}\epsilon^i \otimes \epsilon^j$ and $h^{ij}g_{jk} = \delta^i_k$ where δ^i_k is the Kronecker delta symbol. Show that under a basis transformation: $e_i \to \tilde{e}_i = a^j_i e_j$ with (a^j_i) some $n \times n$ invertible matrix, the numbers h^{ij} transform like the components of a tensor h of type (2,0). (15 points)

Hint: Let the transformation induce $g_{ij} \to \tilde{g}_{ij}$ and $h^{ij} \to \tilde{h}^{ij}$. The aim is to use $h^{ij}g_{jk} = \delta^i_k$ and $\tilde{h}^{ij}\tilde{g}_{jk} = \delta^i_k$ to express $\tilde{h}^{ij}$ in terms of h^{ij} and a^i_j or b^i_j, where (b^i_j) is the inverse of (a^i_j). To do this proceed as follows.

(i) Express g_{ij} in terms of $\tilde{g}_{ij}$ and a^i_j or b^i_j. (ii) Substitute this expression for g_{ij} in $h^{ij}g_{jk} = \delta^i_k$ making sure you use dummy indices different from i, j, k, ℓ, r, s. (iii) Multiply both sides of the resulting expression by $\tilde{h}^{\ell r}a^k_\ell b^s_i$ and sum over repeated indices. (iv) Using $\tilde{h}^{ij}\tilde{g}_{jk} = \delta^i_k$ and the fact that g is a symmetric tensor, you should be able to determine $\tilde{h}^{ij}$ in terms of h^{ij} and a^i_j or b^i_j.

4.c) Show that $h := h^{ij} e_i \otimes e_j$ is a symmetric tensor. (10 points)

Hint: Multiply both sides of $h^{ij} g_{jk} = \delta^i_k$ by $g_{i\ell}$ and try to relabel the dummy indices and use the fact that g is symmetric to show that $h^{ij} = h^{ji}$.