1) Let V and W be a complex vector spaces and $L : V \to W$ be a one-to-one linear operator. Show $L^{-1} : W \to V$ is also a linear operator.
Warning: Do not assume that domain of L is V or that L is onto.

2) Let V be the set of polynomial $p : [-1, 1] \to \mathbb{C}$ of degree not greater than 2, r be an arbitrary element of V, i.e., $r(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ for some $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$, $D : V \to V$ be the differentiation operator, $(Dr)(x) := r'(x) = \alpha_1 + 2\alpha_2 x$, $\mathscr{B} := \{p_0, p_1, p_2\}$ be the monomial basis of V (with $p_i(x) = x^i$ for $i \in \{0, 1, 2\}$) and for all $\gamma \in [0, 1]$, and every $p, q \in V$,

$$\langle p, q\rangle_\gamma := \int_{-1}^{1} (1 + \gamma x)\overline{p(x)}q(x)dx, .$$

2.a) Find the null space and range of D and determine whether it is one-to-one or onto.

2.b) Find the matrix representation of D in the bases $\mathscr{B}$.

2.c) Show that $\langle \cdot, \cdot\rangle_\gamma$ is an inner product on V.

2.d) Apply the Gram-Schmidt process on $\mathscr{B}$ to construct an orthonormal basis $\mathscr{E}$ for the inner-product space $(V, \langle \cdot, \cdot\rangle_\gamma)$.

2.e) Find the matrix representation of D in the basis $\mathscr{E}$.

2.f) Construct the complete orthonormal system of orthogonal projection operators, $\{P_1, P_2, P_3\}$, associated with the basis $\mathscr{E}$, i.e., give an explicit formula for $(P_i r)(x)$ for each $i \in \{0, 1, 2\}$.

3) Let V be the set of 2×2 complex matrices,

$$\mathbf{M}_1 := \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{M}_2 := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{M}_3 := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{M}_1 := \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$\mathscr{B} := \{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\mathbf{A} := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, and for all $\mathbf{L}, \mathbf{M} \in V$,

$$\langle \mathbf{L}, \mathbf{M}\rangle := \text{Trace}(\mathbf{L}^\dagger \mathbf{M}).$$

3.a) Show that $\mathscr{B}$ is a basis of V.

3.b) Find the matrix representation of $\mathbf{A}$ in the basis $\mathscr{B}$. Recall that the result is 4×1 matrix.

3.c) Show that $\langle \cdot, \cdot\rangle$ is an inner product on V.

3.d) Apply the Gram-Schmidt process on $\mathscr{B}$ to construct an orthonormal basis of the inner-product space $(V, \langle \cdot, \cdot\rangle)$.

3.e) Find the matrix representation of $\mathbf{A}$ in the basis you find in part 3.d.

4) Let V be a complex inner-product space. Find complex numbers $\alpha, \beta, \gamma, \delta$ such that for all $a, b \in V$,

$$\langle a, b\rangle = \alpha \parallel a - b \parallel^2 + \beta \parallel a + b \parallel^2 + \gamma \parallel a - i\, b \parallel^2 + \delta \parallel a + i\, b \parallel^2 .$$

Note that this shows that the inner product is uniquely determined by the norm it defines.

5) Let V and W be complex inner-product spaces with inner products $\langle \cdot, \cdot\rangle_V$ and $\langle \cdot, \cdot\rangle_W$, and $U : V \to W$ be a linear operator such that for all $v \in \mathrm{Dom}(U)$, $\langle v, v\rangle_V = \langle Uv, Uv\rangle_W$.

5.a) Show that for all $a, b \in \mathrm{Dom}(U)$, $\langle a, b\rangle_V = \langle Ua, Ub\rangle_W$.

5.b) Show that U is one-to-one.

6) Let V and W be complex inner-product spaces, $U : V \to W$ is a unitary operator and $H : V \to V$ be a Hermitian operator with domain V. Show that $UHU^{-1} : W \to W$ is a Hermitian operator with domain W.

7) Let $\mathcal{E} := \{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of an inner-product space V, $P_i : V \to V$ be defined by $P_i v := \langle e_i, v\rangle e_i$, where $i \in \{1, 2, \cdots, n\}$ and $v \in V$ are arbitrary, and for each $m \in \{1, 2, \cdots, n-1\}$, $\Pi_m := P_1 + P_2 + \cdots + P_m$.

7.a) Determine $\mathrm{Nul}(\Pi_m)$ and $\mathrm{Ran}(\Pi_m)$.

7.b) Show that for all $a \in V$, $\parallel \Pi_m a \parallel^2 = \sum_{i=1}^m |\langle e_i, a\rangle|^2$.

7.c) Show that $\Pi_m : V \to V$ is a projection operator and determine if it is an orthogonal projection operator.

7.d) Show that for all $u \in V$, $\parallel \Pi_m u \parallel \leq \parallel u \parallel$. This is known as Bessel's inequality.

8) Let V be a complex inner-product space and $J, K, L \in \mathcal{G}\ell(V)$. Show that

$$\mathrm{Trace}(JKL) = \mathrm{Trace}(LJK).$$