Math 320: Quiz # 4

Spring 2015

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	Solutions

- You have 50 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- No question are answered during this quiz.

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1) (10 points) Give the definition of the following terms.

a. Square root of a linear operator:

A linear operator S is called a square root of an operator T if $S^2 = T$.

b. Isometry:

An operator $S \in \mathcal{L}(V)$ is called an isometry if
$\|Sv\| = \|v\|$

c. Polar decomposition of a linear operator:

Let $T \in \mathcal{L}(V)$, then $\exists$ an isometry $S \in \mathcal{L}(V)$ s.t. $T = S\sqrt{T^*T}$. This is called the polar decomposition of T.

d. Generalized eigenvector:

Let $T \in \mathcal{L}(V)$ and λ be an eigenvalue of T. A nonzero vector $v \in V$ is called a generalized eigenvector of T corr. to λ if $(T-\lambda I)^i v = 0$ for some $i \in \mathbb{Z}^+$.

e. Algebraic multiplicity of an eigenvalue:

The algebraic multiplicity of an eigenvalue λ of an operator T is $\dim \operatorname{Null}(T-\lambda I)^{\dim V}$

2) (10 points) Let V be an inner-product space and $T \in \mathcal{L}(V)$ be a normal operator. Show that for all $k \in \mathbb{Z}^+$, $\mathrm{Nul}(T^k) = \mathrm{Nul}(T)$.

For $k=1$ it is obvious.

Assume $k \geq 2$.

$v \in Null(T) \Rightarrow T^k v = T^{k-1}(\underbrace{Tv}_{0}) = 0 \Rightarrow v \in Null(T^k)$

So, $Null(T) \subseteq Null(T^k)$ (*)

To prove $Null(T) \supseteq Null(T^k)$, we will use mathematical induction:

(i) For $k=1$ $Null(T^1) \subseteq Null(T)$

(ii) Assume $Null(T^m) \subseteq Null(T)$ for $m \geq 2$ $(k=m)$

We will prove $Null(T^{m+1}) \subseteq Null(T)$ $(k=m+1)$

Let $v \in Null(T^{m+1})$,

$$\begin{aligned} \langle T^* T^m v, T^* T^m v \rangle &= \langle T T^* T^m v, T^m v \rangle \\ &= \langle T^* \underbrace{T^{m+1} v}_{0}, T^m v \rangle \text{ (by normality)} \\ &= 0 \end{aligned}$$

Then, $T^* T^m v = 0$.

$$\begin{aligned} \Rightarrow 0 &= \langle T^* T^m v, T^{m-1} v \rangle \\ &= \langle T^m v, T^m v \rangle \end{aligned}$$

$\Rightarrow T^m v = 0 \Rightarrow v \in Null(T^m) \Rightarrow v \in Null(T)$ by (i).

$\Rightarrow Null(T^{m+1}) \subseteq Null(T)$

By induction, $Null(T^k) \subseteq Null(T)$ for $k \in \mathbb{Z}^+$ (**)

$\therefore$ (*) and (**) imply that $Null(T^k) = Null(T)$ for all $k \in \mathbb{Z}^+$.

3) (10 points) Let V be an inner-product space, $T \in \mathcal{L}(V)$ be a self-adjoint operator, $I \in \mathcal{L}(V)$ be the identity operator, and $\alpha, \beta \in \mathbb{R}$ such that $4\beta - \alpha^2 \geq 0$. Show that $T^2 + \alpha T + \beta I$ is a positive operator.

Let $T \in \mathcal{L}(V)$ be a self-adjoint operator. For $v \in V$,

(i) $\langle (T^2 + \alpha T + \beta I)v, v\rangle = \langle T^2 v, v\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle$ $(\alpha, \beta \in \mathbb{R})$

$$= \underbrace{\langle Tv, Tv\rangle}_{\text{Since } T \text{ is self-adjoint}} + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle$$

$$\geq \|Tv\|^2 + |\alpha|\|Tv\|\|v\| + \beta\|v\|^2$$

(by Cauchy inequality)

$$\geq \underbrace{\left(\|Tv\| - \frac{|\alpha|\|v\|}{2}\right)^2}_{\geq 0} + \underbrace{\left(\beta - \frac{\alpha^2}{4}\right)\|v\|^2}_{\geq 0 \text{ if } 4\beta - \alpha^2 \geq 0.}$$

(ii) $(T^2 + \alpha T + \beta I)^* = (T^2)^* + \bar{\alpha} T^* + \bar{\beta} I$

$$= (T^*)^2 + \alpha T^* + \beta I \quad (\text{since } \alpha, \beta \text{ are real})$$

$$= T^2 + \alpha T + \beta I \quad (T = T^*)$$

So, $T^2 + \alpha T + \beta I$ is self-adjoint.

$\therefore$ (i) and (ii) imply that $T^2 + \alpha T + \beta I$ is a positive operator if $4\beta - \alpha^2 \geq 0$.

4 (10 points) Let V be an inner-product space and $T_1, T_2 \in \mathcal{L}(V)$. Show that T_1 and T_2 have the same singular values if and only if there are isometries $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1 T_2 S_2$.

($\Rightarrow$). Assume that T_1 and T_2 have same singular values $s_1, --s_n$. Then there exist orthonormal bases $(e_1, e_2, --e_n)$, $(f_1 -- f_n)$ and $(\tilde{e}_1, --, \tilde{e}_n)$ & $(\tilde{f}_1, -- \tilde{f}_n)$ s.t.

$$(*) \quad T_1 v = s_1 \langle v, e_1 \rangle f_1 + --+ s_n \langle v, e_n \rangle f_n$$
$$(**) \quad T_2 v = s_1 \langle v, \tilde{e}_1 \rangle \tilde{f}_1 + --+ s_n \langle v, \tilde{e}_n \rangle \tilde{f}_n \quad \text{for any } v \in V.$$

Let $S_1, S_2 \in \mathcal{L}(V)$ be defined by,

$$S_1(\tilde{f}_i) = f_i$$
$$S_2(e_i) = \tilde{e}_i$$

Since S_1, S_2 map orthonormal lists to orthonormal lists, they are isometries. Using $(**)$, we find

$$T_2 S_2 v = s_1 \langle S_2 v, \tilde{e}_1 \rangle \tilde{f}_1 + --+ s_n \langle S_2 v, \tilde{e}_n \rangle \tilde{f}_n$$
$$= s_1 \langle v, S_2^* \tilde{e}_1 \rangle \tilde{f}_1 + ---+ s_n \langle v, S_2^* \tilde{e}_n \rangle \tilde{f}_n$$
($S_2^* = S_2^{-1}$, because S_2 is an isometry)
$$= s_1 \langle v, e_1 \rangle \tilde{f}_1 + --+ s_n \langle v, e_n \rangle \tilde{f}_n$$

$$\Rightarrow S_1 T_2 S_2 v = s_1 \langle v, e_1 \rangle S_1(\tilde{f}_1) + --+ s_n \langle v, e_n \rangle S_n(\tilde{f}_n)$$
$$= s_1 \langle v, e_1 \rangle f_1 + --+ s_n \langle v, e_n \rangle f_n$$
$$= T_1 v \quad (\text{by } (*)), \text{ for any } v \in V.$$

Thus, $S_1 T_2 S_2 = T_1$.

($\Leftarrow$) Let $S_1, S_2 \in \mathcal{L}(V)$ be isometries s.t. $T_1 = S_1 T_2 S_2$. for $T_1, T_2 \in \mathcal{L}(V)$, Then $T_1^* T_1 = S_2^* T_2^* \underbrace{S_1^* S_1}_{I} T_2 S_2 = S_2^{-1} T_2^* T_2 S_2$ ($S_2^{-1} = S_2^*$ since S_2 is an isometry)

Let λ be an eigenvalue of $T_1^* T_1$ with eigenvector v.

$T_1^* T_1 v = S_2^{-1} T_2^* T_2 S_2 v = \lambda v \Rightarrow T_2^* T_2 S_2 v = \lambda S_2 v$ ($S_2 v \neq 0$, because $v \neq 0$)

$\Rightarrow \lambda$ is an eigenvalue of $T_2^* T_2$. Also $\dim \text{Null}(T_1^* T_1 - \lambda I) = \dim \text{Null}(T_2^* T_2 - \lambda I)$ since S_2 is injective. (Same can be done for T_2 with isometries S_1^*, S_2^*).

Hence, singular values of $T_1^* T_1$ and $T_2^* T_2$ are the same.