Math 320: Quiz # 1

Spring 2015

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	Solutions

- You have 50 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- No question are answered during this quiz.

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1 **(10 points)** Give the definition of the following terms.

a) Subspace of a vector space:
A subset $\mathcal{U}$ of a vector space is a subspace if the following hold
(i) $0 \in \mathcal{U}$
(ii) $u, v \in \mathcal{U} \Rightarrow u+v \in \mathcal{U}$
(iii) $a \in \mathbb{F}$, $u \in \mathcal{U} \Rightarrow au \in \mathcal{U}$.

b) Sum of two subspaces of a vector space:
Let $\mathcal{U}_1, \mathcal{U}_2$ be subspaces of a vector space. The sum of $\mathcal{U}_1$ and $\mathcal{U}_2$ is
$\mathcal{U}_1 + \mathcal{U}_2 := \{u_1 + u_2 : u_1 \in \mathcal{U}_1, u_2 \in \mathcal{U}_2\}$

c) Linearly independent list $(v_1, v_2, \cdots, v_m)$:
A list $(v_1, \ldots v_m)$ of vectors in a vector space is called linearly independent if for $a_1, a_2 \ldots a_m \in \mathbb{F}$ $a_1v_1 + \ldots + a_mv_m = 0$ only when $a_1 = a_2 = \ldots = 0$.

d) Linearly independent subset of a vector space:
A subset S of a vector space is linearly independent if $S \neq \{0\}$ and every list in S is a linearly independent list.

e) Dimension of a finite-dimensional vector space
The number of elts in a basis of a vector space is called the dimension of this vector space.

2 (10 points) Let U and W be subspaces of a vector space V such that $U \cap W = \{0\}$ and $V = U + W$. Show that $V = U \oplus W$.

Suppose $V = U + W$ and $U \cap W = \{0\}$. To show $V = U \oplus W$, it is enough to show that for $u \in U, w \in W$ $u + v = 0 \Rightarrow u = v = 0$.

$u + w = 0 \Rightarrow u = -w$, i.e. $-w \in U$

Since W is a subspace $-w \in W$.

Then, $-w \in U \cap W$. However, we assumed $U \cap W = \{0\}$

Hence, $u = -w = 0 \Rightarrow u = w = 0$

$\Rightarrow V = U \oplus W$.

3 (10 points) Give a proof or a counterexample for the following statement. "if U, W_1, and W_2 are subspaces of a vector space V such that $V = U \oplus W_1$ and $V = U \oplus W_2$, then $W_1 = W_2$."

The above statement is not true, because if we take

$V = \mathbb{R}^2$, $\quad U = \{(x,x) : x \in \mathbb{R}\}$, $\quad W_2 = \{(0,y) : y \in \mathbb{R}\}$
$W_1 = \{(z,0) : z \in \mathbb{R}\}$.

$V = U \oplus W_1 = U \oplus W_2$, however $W_1 \neq W_2$.

4 (10 points) Prove that the vector space $\mathbb{R}^\infty$ of real sequences $(a_1, a_2, a_3, \cdots)$ is infinite-dimensional.

To show that $\mathbb{R}^\infty$ is infinite dimensional it is enough to find a linearly independent list of n vectors for each $n \in \mathbb{Z}^+$.

Let
$$e_1 = (1,0,0,\ldots)$$
$$e_2 = (0,1,0,0\ldots)$$
$$\vdots$$
$$e_n = (0,0,\ldots,0,\underset{n^{th}\text{ coordinate}}{1},0\ldots)$$

For each $n \in \mathbb{Z}^+$, $(e_1, \ldots e_n)$ is a linearly independent list in $\mathbb{R}^\infty$.
Because, for $a_1, a_2 \ldots a_n \in \mathbb{R}$

$$a_1e_1 + \ldots + a_ne_n = 0 \Leftrightarrow (a_1, a_2, \ldots, a_n, 0, 0\ldots) = (0,0,\ldots\ldots)$$
$$\Leftrightarrow a_1 = a_2 = a_3 = \ldots = a_n = 0.$$