Math 320: Midterm Exam # 2

Spring 2015

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	Solutions

- You have 2 hours.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 1 (20 points) Let V be an inner-product space, $T \in \mathcal{L}(V)$ be a self-adjoint operator, $I \in \mathcal{L}(V)$ be the identity operator, and $S := T^2 + \alpha T + \beta I$, where $\alpha, \beta \in \mathbb{R}$. Show that if $\alpha^2 \leq 4\beta$, then S is a positive operator.

(i) S is self adjoint, because

$$
\begin{aligned}
S^* &= (T^2)^* + (\alpha T)^* + (\beta I)^* \\
&= (T^*)^2 + \bar{\alpha} T^* + \bar{\beta} I \\
&= T^2 + \alpha T + \beta I \quad (\text{since } \alpha, \beta \in \mathbb{R} \text{ and } T \text{ is self-adjoint}) \\
&= S
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\langle Sv, v\rangle &= \langle T^2 v, v\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle \\
&= \langle Tv, Tv\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle \quad (T = T^*) \\
&\geq \|Tv\|^2 + |\alpha|\,\|Tv\|\,\|v\| + \beta\|v\|^2 \quad (\text{by Cauchy Inequality}) \\
&\geq \underbrace{\left(\|Tv\| + \frac{|\alpha|\,\|v\|}{2}\right)^2}_{\geq 0} + \underbrace{\left(\beta - \frac{\alpha^2}{4}\right)\|v\|^2}_{\geq 0 \text{ if } 4\beta - \alpha^2 \geq 0}
\end{aligned}
$$

Thus, S is a positive operator if $4\beta - \alpha^2 \geq 0$.

Problem 2 (20 points) Let V be a finite-dimensional complex inner-product space and $S \in \mathcal{L}(V)$. Show that S is an isometry if and only if there is an orthonormal basis of V consisting of the eigenvectors of S and that the eigenvalues λ of S are unimodular, i.e., $|\lambda| = 1$.

($\Rightarrow$) Suppose $S \in \mathcal{L}(V)$ is an isometry. Then S is a normal operator. By the complex spectral thm there exists an orthonormal basis $(e_1, \dots, e_n)$ of V consisting of eigenvectors of S.

Let λ_i denote the eigenvalue corresponding to eigenvector e_i, then

$$Se_i = \lambda_i e_i \qquad i \in \{1, \dots, n\}$$

$$\Rightarrow \|Se_i\| = |\lambda_i| \|e_i\|$$

$$\Rightarrow \|e_i\| = \|e_i\| |\lambda_i| \quad (\text{because } S \text{ is an isometry})$$

$$\Rightarrow |\lambda_i| = 1$$

($\Leftarrow$) Suppose there is an orthonormal basis $(e_1, \dots e_n)$ of V s.t. $Se_i = \lambda_i e_i$ and $|\lambda_i| = 1$, where $\lambda_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$

Using this orthonormal basis we can write any $v \in V$ as:

$v = \alpha_1 e_1 + \dots + \alpha_n e_n$ s.t.

$\|v\|^2 = |\alpha_1|^2 + \dots + |\alpha_n|^2$, $\alpha_1 \dots \alpha_n \in \mathbb{C}$.

$$\Rightarrow Sv = \alpha_1 \underbrace{Se_1}_{\lambda_1 e_1} + \dots + \alpha_n \underbrace{Se_n}_{\lambda_n e_n}$$

$$\Rightarrow \|Sv\|^2 = \underbrace{|\lambda_1|^2}_{1} |\alpha_1|^2 + \dots + \underbrace{|\lambda_n|^2}_{1} |\alpha_n|^2$$

$$= \|v\|^2$$

Hence, S is an isometry.

Problem 3 (20 points) Let V a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Show that every root of the minimal polynomial for T is an eigenvalue of T.

Let $p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$ be the minimal polynomial of $T \in \mathcal{L}(V)$.

$\lambda \in \mathbb{F}$ is a root of $p \Rightarrow p(z) = (z-\lambda)\, q(z)$

where q is a monic polynomial s.t. $\deg p > \deg q$.

By the definition of minimal polynomial,

$p(T) = (T - \lambda I)\, q(T) = 0$

and since $\deg p > \deg q$ there exists $v \in V$ such that $q(T)v \neq 0$. (Otherwise q would be the minimal poly)

$(T-\lambda I)\, q(T) v = 0$.

This implies that $q(T)v$ is an eigenvector of T with eigenvalue of λ. So, every root of the minimal poly. for T is an eigenvalue of T.

Problem 4 (20 points) Let V be a finite-dimensional complex inner-product space, $T \in \mathcal{L}(V)$ be a normal operator, and p be the minimal polynomial for T. Show that p does not have repeated roots.

By complex spectral thm, there exists a basis $(e_1, -- e_n)$ of V consisting of eigenvectors of T.
Let $\lambda_1, ---, \lambda_m$ be distinct eigenvectors of T.
Then, $(T-\lambda_1 I) ---- (T-\lambda_m I) e_j = 0$ for $j \in \{1 -- n\}$
Since $(e_1, ---, e_n)$ is a basis of V,
$(T-\lambda_1 I) -- (T-\lambda_m I) v = 0$ for any $v \in V$.
If we define a polynomial q as :
$q(z) = (z-\lambda_1) --- (z-\lambda_m)$, then $q(T) = 0$.
By the defn of minimal poly, p should divide q.
Since q has no repeated roots, p does not have repeated roots also.

Problem 5 Let $N \in \mathcal{L}(\mathbb{F}^4)$ be defined by $N(x_1, x_2, x_3, x_4) := (2x_2, 3x_3, -4x_4, 0)$ and $I \in \mathcal{L}(\mathbb{F}^4)$ be the identity operator.

5.a (5 points) Show that N is nilpotent.

$$N(x_1, x_2, x_3, x_4) = (2x_2, 3x_3, -4x_4, 0)$$

$$N^2(x_1, x_2, x_3, x_4) = (6x_3, -12x_4, 0, 0)$$

$$N^3(x_1, x_2, x_3, x_4) = (-24x_4, 0, 0, 0)$$

$$N^4(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$$

Hence, N is nilpotent.

5.b (15 points) Let T be an arbitrary square root of $N + I$. Give an explicit expression for $T(x_1, x_2, x_3, x_4)$.

Consider the polynomial $I + a_1 N + a_2 N^2 + a_3 N^3$, where $a_1, a_2, a_3 \in \mathbb{F}$.

We want to solve

$$I + N = (I + a_1 N + a_2 N^2 + a_3 N^3)^2$$
$$= I + 2a_1 N + (2a_2 + a_1^2) N^2 + (2a_3 + 2a_1 a_2) N^3.$$

This eqn. will be satisfied, if we choose:

$$\left.\begin{aligned} a_1 &= \frac{1}{2} \\ 2a_2 + a_1^2 &= 0 \\ 2a_3 + 2a_1 a_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= \frac{1}{2} \\ a_2 &= -\frac{1}{8} \\ a_3 &= \frac{1}{16} \end{aligned}$$

Define, $T = I + \frac{1}{2} N - \frac{1}{8} N^2 + \frac{1}{16} N^3$

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - \frac{3}{4} x_3 - \frac{3}{2} x_4,\ x_2 + \frac{3}{2} x_3 + \frac{3}{2} x_4,\ x_3 - 2x_4,\ x_4)$$

and $T^2(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, x_2 + 3x_3; x_3 - 4x_4, x_4)$
$= (I + N)(x_1, x_2, x_3, x_4)$.

So, T is a square root of $I + N$.