Solutions to Math 320
Midterm Exam 1

Problem 1 Let V and W be vector spaces over $\mathbb{F}$, $D \subseteq V$, $L : V \to W$ be a linear map with domain D, B be a subset of W, and A be the inverse image of B under L, i.e., $A := L^{-1}(B)$. Prove or disprove the following statements.

1.a (10 points) If A is a subspace of D, then B is a subspace of W.

This is false, because for example take

$V = W = \mathbb{R}$, $L =$ zero map, $B = \{0, 1\}$

Then $D = \mathbb{R}$ and $A = L^{-1}(B) = \{x \in \mathbb{R} \mid Lx \in B\}$

$\forall x \in \mathbb{R}$, $Lx = 0 \in B \Rightarrow A = L^{-1}(B) = \mathbb{R}$ which is a subspace of $\mathbb{R}$. But B is not a subspace of $\mathbb{R}$, because $1 \in B$ but $2 = 2 \cdot 1 \notin B$.

1.b (10 points) If B is a subspace of W, then A is a subspace of D.

This is true and we prove it.

- $L0_V = 0_W \in B \Rightarrow 0_V \in L^{-1}(B) = A$ (1)

- $\forall a_1, a_2 \in A$ and $\forall \alpha \in \mathbb{F}$

$$A = L^{-1}(B) := \{a \in D \mid La \in B\} \subseteq D$$

$\Downarrow$

$\exists b_1, b_2 \in B$, $La_1 = b_1$ and $La_2 = b_2$

$A \subseteq D \Rightarrow a_1, a_2 \in D \Rightarrow a_1 + \alpha a_2 \in D$

(D is a subspace of V)

and $L(a_1 + \alpha a_2) = La_1 + \alpha La_2 = b_1 + \alpha b_2$

B is a subspace of $W \Rightarrow b_1 + \alpha b_2 \in B$

$\Downarrow$

$a_1 + \alpha a_2 \in L^{-1}(B) = A$ (2)

(1) & (2) $\Rightarrow$ A is a subspace of V.

Problem 2 Let U, V and W be vector spaces over $\mathbb{F}$, U and V are finite-dimensional, $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$, and $\tilde{S} : \mathrm{Ran}(T) \to W$ be the restriction of S to $\mathrm{Ran}(T)$, i.e., for all $v \in \mathrm{Ran}(T)$, $\tilde{S}v := Sv$.

2.a (3 points) Show that $\mathrm{Nul}(\tilde{S})$ is a subspace of $\mathrm{Nul}(S)$.

$\mathrm{Nul}\,\tilde{S} \leq \mathrm{Ran}(T) \Rightarrow \forall v \in \mathrm{Nul}\,\tilde{S},\ v \in \mathrm{Ran}(T) \Rightarrow$
$Sv = \tilde{S}v = 0 \Rightarrow v \in \mathrm{Nul}(S) \Rightarrow \mathrm{Nul}(\tilde{S}) \subseteq \mathrm{Nul}(S)$
$\mathrm{Nul}(\tilde{S})$ is a subspace of $\mathrm{Ran}(T) \Rightarrow$ it is a subspace of $V \Rightarrow \mathrm{Nul}(\tilde{S}) \leq \mathrm{Nul}(S)$.

2.b (3 points) Show that $\mathrm{Ran}(\tilde{S}) = \mathrm{Ran}(ST)$.

$\forall w \in \mathrm{Ran}(\tilde{S}) \Leftrightarrow (\exists v \in \mathrm{Ran}(T),\ \tilde{S}v = w \Rightarrow Sv = w$
$\&\ v \in \mathrm{Ran}(T) \Rightarrow \exists u \in U,\ v = Tu$ $\hookrightarrow w = S(Tu) = STu$
$\Rightarrow w \in \mathrm{Ran}(ST) \Rightarrow \mathrm{Ran}(\tilde{S}) \subseteq \mathrm{Ran}(ST)$. ①
$\forall w \in \mathrm{Ran}(ST),\ \exists u \in U,\ STu = w \Rightarrow S(Tu) = w \Rightarrow$
$Tu \in \mathrm{Ran}(T) \hookrightarrow S(Tu) = \tilde{S}(Tu) \hookrightarrow w = \tilde{S}(Tu) \Rightarrow$
$w \in \mathrm{Ran}(\tilde{S}) \hookrightarrow \mathrm{Ran}(ST) \subseteq \mathrm{Ran}(\tilde{S})$ ② ① & ② $\Rightarrow$

2.c (14 points) Use parts a and b of this problem together with the dimension formula for T and $\tilde{S}$ to show that $\dim \mathrm{Nul}(ST) \leq \dim \mathrm{Nul}(S) + \dim \mathrm{Nul}(T)$.

, ST

$\dim \mathrm{Nul}(T) + \dim \mathrm{Ran}(T) = \dim(U)$ ③
$\dim \mathrm{Nul}(\tilde{S}) + \dim \mathrm{Ran}(\tilde{S}) = \dim(\mathrm{Ran}(T))$ ④
$\dim \mathrm{Ran}(\tilde{S}) = \dim \mathrm{Ran}(ST)$ ⑤

$\dim \mathrm{Nul}(ST) + \dim \mathrm{Ran}(ST) = \dim(U)$ ⑥

4 & 5 $\Rightarrow$ $\dim \mathrm{Nul}(T) + \dim \mathrm{Nul}(\tilde{S}) + \dim \mathrm{Ran}(ST) = \dim(U)$
$\quad = \dim \mathrm{Nul}(ST) + \dim \mathrm{Ran}(ST)$ ⑥

$\Rightarrow \dim \mathrm{Nul}(T) + \dim \mathrm{Nul}(\tilde{S}) = \dim \mathrm{Nul}(ST)$

$\dim \mathrm{Nul}(\tilde{S}) \leq \dim \mathrm{Nul}(S)$ ← by (2.a)

$\Rightarrow \dim \mathrm{Nul}(ST) \leq \dim \mathrm{Nul}(T) + \dim \mathrm{Nul}(S)$ ☒

Problem 3 Let V be the vector space of upper-triangular complex 2×2 matrices and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be defined by

$$\forall \mathbf{A}, \mathbf{B} \in V, \quad \langle \mathbf{A}, \mathbf{B} \rangle := \mathrm{tr}(\mathbf{A}\overline{\mathbf{B}}^T),$$

where for every $\mathbf{M} = [M_{ij}] \in V$, $\mathrm{tr}(\mathbf{M})$ is the sum of the diagonal entries of $\mathbf{M}$, i.e., $\mathrm{tr}(\mathbf{M}) = M_{11} + M_{22}$, and $\overline{\mathbf{A}}^T$ is the transpose of the complex-conjugate of $\mathbf{M}$; $\overline{\mathbf{M}}^T = [\overline{M_{ji}}]$.

3.a (10 points) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.

$\mathbb{A} = [A_{ij}]$, $\mathbb{B} = [B_{ij}]$, $\overline{\mathbb{B}}^T = [\overline{\mathbb{B}}^T_{ij}]$ $\Rightarrow$

$$[\mathbb{A}\overline{\mathbb{B}}^T]_{ij} = \sum_{k=1}^{2} A_{ik} \overline{\mathbb{B}}^T_{kj} = \sum_{k=1}^{2} A_{ik} \overline{B_{jk}}$$

$$\Rightarrow \langle \mathbb{A}, \mathbb{B} \rangle = \sum_{i=1}^{2} (\mathbb{A}\overline{\mathbb{B}}^T)_{ii} = \sum_{i,k=1}^{2} A_{ik} \overline{B_{ik}} \quad (*)$$

- $\langle \mathbb{A}, \mathbb{A} \rangle = \sum_{i,k=1}^{2} A_{ik} \overline{A_{ik}} = \sum_{i,k=1}^{2} |A_{ik}|^2 \in \mathbb{R}^+ \cup \{0\}$

- $\langle \mathbb{A}, \mathbb{A} \rangle = 0 \iff \forall i,j \in \{1,2\}, \ |A_{ik}|^2 = 0 \Leftarrow A_{ik} = 0$

$\iff \mathbb{A} = 0$ (1)

- $$\overline{\langle \mathbb{A}, \mathbb{B} \rangle} = \overline{\sum_{i,k=1}^{2} A_{ik} \overline{B_{ik}}} = \sum_{i,k=1}^{2} \overline{A_{ik}} B_{ik}$$

$$= \sum_{i,k=1}^{2} B_{ik} \overline{A_{ik}} = \langle \mathbb{B}, \mathbb{A} \rangle \quad (2)$$

- $\forall \mathbb{A}_1, \mathbb{A}_2, \mathbb{B} \in V$, $\forall \alpha \in \mathbb{C}$,

$$\overset{*}{\Rightarrow} \langle \mathbb{A}_1 + \alpha \mathbb{A}_2, \mathbb{B} \rangle = \sum_{i,k=1}^{2} [\mathbb{A}_1 + \alpha \mathbb{A}_2]_{ik} \overline{B_{ik}}$$

$$= \sum_{i,k=1}^{2} (A_{1ik} + \alpha A_{2ik}) \overline{B_{ik}}$$

$$= \sum_{i,k=1}^{2} A_{1ik} \overline{B_{ik}} + \alpha \sum_{i,k=1}^{2} A_{2ik} \overline{B_{ik}}$$

$$= \langle \mathbb{A}_1, \mathbb{B} \rangle + \alpha \langle \mathbb{A}_2, \mathbb{B} \rangle \quad (3)$$

(1), (2), (3) $\Rightarrow$ $\langle \cdot, \cdot \rangle$ is an inner product on V.

3.b (10 points) Find the general form of a unit element of the inner-product space $(V, \langle \cdot, \cdot \rangle)$ which is orthogonal to both of the following matrices.

$$\mathbf{M}_1 := \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}, \quad \mathbf{M}_2 := \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}.$$

$$\mathbb{A} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$$\mathbb{A}\,\overline{\mathbb{M}}_1^T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \overline{\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}}^T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} a-ib & b \\ -id & d \end{bmatrix}$$

$$\mathbb{A}\,\overline{\mathbb{M}}_2^T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \overline{\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}}^T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} a+ib & b \\ id & d \end{bmatrix}$$

$$0 = \langle \mathbb{A}, \mathbb{M}_1 \rangle = tr(\mathbb{A}\,\overline{\mathbb{M}}_1^T) = a - ib + d$$

$$0 = \langle \mathbb{A}, \mathbb{M}_2 \rangle = tr(\mathbb{A}\,\overline{\mathbb{M}}_2^T) = a + ib + d$$

$$\Rightarrow \quad a + d = 0 \Rightarrow d = -a, \quad b = 0$$

$$\Rightarrow \quad \mathbb{A} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$$

$$1 = \langle \mathbb{A}, \mathbb{A} \rangle = tr(\mathbb{A}\,\overline{\mathbb{A}}^T) = tr\left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} \bar{a} & 0 \\ 0 & -\bar{a} \end{bmatrix}\right)$$

$$= tr\begin{bmatrix} a\bar{a} & 0 \\ 0 & a\bar{a} \end{bmatrix} = 2a\bar{a} = 2|a|^2$$

$$\Rightarrow \quad |a| = \frac{1}{\sqrt{2}} \Rightarrow a = \frac{1}{\sqrt{2}} e^{i\varphi} \quad \text{for some } \varphi \in [0, 2\pi)$$

$$\text{So} \quad \mathbb{A} = \begin{bmatrix} \frac{e^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{e^{i\varphi}}{\sqrt{2}} \end{bmatrix}, \quad \varphi \in [0, 2\pi)$$

Problem 4 (20 points) Let V be a complex finite-dimensional vector space, $\dim(V) > 0$, and $L \in \mathcal{L}(V)$. Show that L has at least one eigenvalue.

$\lambda \in \mathbb{C}$ is an eigenvalue of L $\iff$ $Nul(L - \lambda I) \neq \{0\}$

$\iff$ $L - \lambda I$ is not invertible.

(**)

$\dim V > 0 \Rightarrow \exists v \in V \setminus \{0\}$. Consider the list

$(v, Lv, L^2 v, \dots, L^n v)$ where $n := \dim V$

This is not linearly-independent because its length is larger than n. $\Rightarrow \exists \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that not all of α_i's are zero and

$$\alpha_0 v + \alpha_1 L v + \alpha_2 L^2 v + \dots + \alpha_n L^n v = 0$$

let $m \in \mathbb{Z}^+$ be the largest positive integer such that $\alpha_m \neq 0$. Then $\underbrace{\alpha_m L^m v + \alpha_{m-1} L^{m-1} v + \dots + \alpha_1 L v + \alpha_0 v = 0}_{p(L)v = 0 \quad (*)}$

where $p(x) := \alpha_m x^m + \alpha_{m-1} x^{m-1} + \dots + \alpha_1 x + \alpha_0$ is a polynomial of degree $m \geq 1$ $\Rightarrow$ it has m complex roots $\lambda_1, \lambda_2, \dots, \lambda_m$ $\Rightarrow$

$$p(x) = \alpha_m (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_m)$$

$$\Rightarrow \quad p(L) = \alpha_m (L - \lambda_1 I)(L - \lambda_2 I) \dots (L - \lambda_m I)$$

If L has no eigenvalues $L - \lambda_i I$ is invertible $\forall i \in \{1, \dots, m\}$

$\Rightarrow$ $p(L)$ is invertible $\Rightarrow$ $Nul(p(L)) = \{0\}$

$*$ $\hookrightarrow$ $v = 0$ which is a contradiction to (**)

$\Rightarrow$ L has an eigenvalue. ☒

Problem 5 (20 points) Let V be a vector space over $\mathbb{F}$ and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same point spectrum, i.e., every eigenvalue of ST is an eigenvalue of TS and vice versa.

Let α be an eignvalue of ST $\Rightarrow$ $\exists v \in V \setminus \{0\}$

$STv = \alpha v$

$\exists$ 2 possibilities:

1) $Tv = 0 \Rightarrow STv = 0 \Rightarrow \alpha v = 0 \Rightarrow \alpha = 0$
$v \neq 0$

$\Downarrow$

T is not invertible

By contradiction suppose that TS is invertible
$\Rightarrow \forall w \in V$, if $Sw = 0 \Rightarrow TSw = 0 \Rightarrow w = 0 \Rightarrow$
$Null(S) = \{0\} \Rightarrow$ S is invertible $\Rightarrow$ S is onto
$v \in V = Ran(S) \Rightarrow$ $\exists a \in V$, $Sa = v$
$v \neq 0 \Rightarrow a \neq 0$

$\Rightarrow TSa = Tv = 0 \Rightarrow 0 = \alpha$ is an eignvalue of TS.

2) $Tv \neq 0 \Rightarrow S(Tv) = \alpha v$
$\Rightarrow TS(Tv) = T(\alpha v) = \alpha Tv$
$Tv \neq 0 \Rightarrow \alpha$ is an eignvalue of TS.

So every eignvalue of ST is an eignvalue of TS. The converse follows by exchanging the role of S and T in the above argument.