MATH 320 - LINEAR ALGEBRA
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SPRING 2005, Quiz #5, Solutions by Turker Ozsari

You may not get credit unless you show all your work and write neatly!
You have 30 minutes.

(1) X is a complex inner product space. Let T be a normal operator on X. Show that there is a linear operator S on X such that $S^2 = T$. (10 Points)
(Hint: By Spectral Theorem, you can write $T = \sum_{i=1}^{m} \lambda_i P_i$, where P_i are mutually orthogonal projections, λ_i are distinct eigenvalues of T, then define an S).
Proof. Define $\xi_i := (\lambda_i)^{1/2}$ and $S := \sum_{i=1}^{m} \xi_i P_i$. Then,
$S^2 = (\sum_{i=1}^{m} \xi_i P_i)(\sum_{j=1}^{m} \xi_j P_j) = \sum_{i,j=1}^{m} \xi_i \xi_j P_i P_j$, since P_i's are mutually orthogonal projections we have $P_i P_j = 0$ for $i \neq j$ and $P_i^2 = P_i$ for each i.
Hence, $S = \sum_{i=1}^{m} (\xi_i)^2 P_i^2 = \sum_{i=1}^{m} \lambda_i P_i = T$.
(2) Show that the linear operator T defined by $T(z_1, z_2) := (z_2, 0)$ on $\mathbb{F}^2$ does not have a square root. (10 Points)
(Hint: Consider the matrix form of T and assume it has a square root to arrive a contradiction.)
Proof. Using the standard basis $\{(1,0),(0,1)\}$, we have the matrix of T as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Assume T has a square root S which has the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, we must have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, since $S^2 = T$.
Then, $a^2 + bc = 0, b(a+d) = 1, c(a+d) = 0, bc + d^2 = 0$. From these equations, we can easliy get the contradiction that $a + d \neq 0$ and $a + d = 0$. By contradiction, T has not a square root.